CLOSED-FORM SOLUTIONS FOR THE RESPONSE OF LINEAR SYSTEMS TO FULLY NONSTATIONARY EARTHQUAKE EXCITATION

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ABSTRACT: New explicit closed-form solutions are derived for the evolutionary correlation and power spectral–density (PSD) matrices characterizing the nonstationary response of linear elastic, both classically and nonclassically damped, multi-degree-of-freedom (MDOF) systems subjected to a fully nonstationary earthquake ground motion process. The newly developed earthquake ground motion model considered represents the temporal variation of both the amplitude and the frequency content typical of real earthquake ground motions. To illustrate the analytical results obtained, a three-dimensional unsymmetrical building equipped with viscous bracings is considered with a single-component ground motion acting obliquely with respect to the building principal directions. These new analytical solutions for structural response statistics are very useful in gaining more physical insight into the nonstationary response behavior of linear dynamic systems subjected to realistic stochastic earthquake ground motion models. Furthermore, the evolution in time of the cross-modal correlation coefficients is examined and compared with the classical stationary solution for white-noise ground motion excitation. The effects of cross-modal correlations on various mean-square response quantities also are investigated using the analytical solutions obtained.

INTRODUCTION

The mode superposition method is recognized generally as a very efficient method for evaluating the dynamic response of viscously damped linear elastic structural systems. When damping is of the form specified by Caughey and O'Kelly (1965), the natural modes of vibration of the system are real-valued and identical to those of the associated undamped system. Systems with this form of damping are said to be classically damped and the classical mode superposition method may be applied to solve for the dynamic response of the system. For systems that do not satisfy the Caughey-O'Kelly condition, the second-order differential equation of motion is recast into the first-order state-space format and the corresponding first-order complex-valued eigenmodes are obtained. Such systems are said to be nonclassically damped and their response may be evaluated by a generalization of the mode superposition method due to Foss (1958). A comprehensive review of complex modal analysis of nonclassically damped linear systems was given by Veletsos and Ventura (1986). The necessity of performing the complex eigenvalue analysis of a large dynamic system may represent an obstacle for practical application of the method. A subspace mode superposition procedure was developed by Mau (1988) to approximate efficiently this complex modal analysis.

A detailed presentation of complex modal analysis for non-classically damped dynamic systems can be found in standard textbooks such as Hurty and Rubinstein (1964) and Lin (1967). Roberts and Spanos (1990) also provide a description of this method and its application in random vibration theory. State-space and complex modal analysis have been used extensively in the past for calculating the second-order cumulants of the response of dynamic systems to stochastic excitations (DebChaudhury and Gasparini 1980; Singh 1980; Wen 1980; Spanos 1983; Igusa et al. 1984; Pradlwarter and Li 1991; Igusa 1992). Recently, this method has been generalized for the computation of the cumulants of any order of the nonstationary response of linear systems when subjected to delta correlated random excitations (Lutes 1986; Lutes and Chen 1992; Papadimitriou and Lutes 1994).

The earthquake ground motion model used in this paper belongs to the family of Gaussian sigma-oscillatory processes. For linear multi-degree-of-freedom (MDOF) systems, the response vector process remains Gaussian and is defined completely by its second-order moments [i.e., correlation matrix or power spectral–density (PSD) matrix]. Thus, new explicit closed-form solutions for the evolutionary (time-varying) correlation and PSD matrices of the response vector to the nonstationary earthquake ground motion model are derived using complex modal analysis. The time-varying cross-modal correlation coefficients also are investigated to assess the effects of neglecting the cross-modal correlations in computing mean-square response functions. The case of a three-dimensional unsymmetrical building with viscous bracings is used to illustrate the application of the explicit closed-form solutions presented here and to gain better insight into the response of linear MDOF systems subjected to a fully nonstationary earthquake ground motion model.

STOCHASTIC EARTHQUAKE GROUND MOTION MODEL

Recently, a stochastic earthquake ground motion model nonstationary in both amplitude and frequency content was proposed as a sigma-oscillatory process (Conte and Peng 1997; Peng 1996). This earthquake ground acceleration model $\ddot{U}_g(t)$ is defined as the sum of a finite number of pairwise independent, uniformly modulated Gaussian processes. Thus

$$\ddot{U}_g(t) = \sum_{k=1}^{p} X_k(t) = \sum_{k=1}^{p} A_k(t) S_k(t) \tag{1}$$

where p = number of component processes; $A_k(t)$ = time modulating function of the kth subprocess or component process $X_k(t)$; and $S_k(t)$ = kth Gaussian stationary process. The time modulating function $A_k(t)$ is defined as

$$A_k(t) = \alpha_k(t - \zeta_k)^{\beta_k} e^{-\gamma_k(t-\zeta_k)} H(t - \zeta_k) \tag{2}$$

where α_k and γ_k = positive constants; β_k = a positive integer; ζ_k = "arrival time" of the kth subprocess $X_k(t)$; and $H(t)$ = Heaviside unit-step function. The kth zero-mean, stationary Gaussian process $S_k(t)$ is characterized by its autocorrelation function

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$$R_{S_k S_k}(\tau) = e^{-\nu_k|\tau|}\cos(\eta_k \tau) \quad (3)$$

and the corresponding PSD function

$$\Phi_{S_k S_k}(\omega) = \frac{\nu_k}{2\pi}\left[\frac{1}{\nu_k^2 + (\omega + \eta_k)^2} + \frac{1}{\nu_k^2 + (\omega - \eta_k)^2}\right] \quad (4)$$

where ν_k and η_k = two free parameters representing the frequency bandwidth and the predominant or central frequency of the process $S_k(t)$, respectively. It can be shown (Conte and Peng 1997) that the mean-square function of the foregoing ground acceleration model can be expressed as

$$E[|\dot{U}_g(t)|^2] = \int_{-\infty}^{\infty}\sum_{k=1}^{p}|A_k(t)|^2\Phi_{S_k S_k}(\omega)d\omega = \sum_{k=1}^{p}|A_k(t)|^2 \quad (5)$$

and the corresponding evolutionary (time-varying) PSD function is given by

$$\Phi_{\dot{U}_g\dot{U}_g}(t, \omega) = \sum_{k=1}^{p}|A_k(t)|^2\Phi_{S_k S_k}(\omega) \quad (6)$$

The evolutionary PSD (EPSD) function gives the time-frequency distribution of the earthquake ground acceleration process.

The next sections are devoted to the nonstationary response analysis of linear MDOF systems subjected to the foregoing fully nonstationary earthquake ground motion model. For the application examples presented at the end, the foregoing earthquake model has been calibrated against two well-known actual earthquake records. The first one is the S00E (N–S) component of the Imperial Valley earthquake of May 18, 1940, recorded at the El Centro site. The second one is the N00W (N–S) component of the San Fernando earthquake of February 9, 1971, recorded at the Orion Blvd. site.

STATE-SPACE FORMULATION OF EQUATIONS OF MOTION OF LINEAR MDOF SYSTEMS

The general equations of motion for an n-DOF linear system are, in matrix form

$$M\ddot{U}(t) + C\dot{U}(t) + KU(t) = PF(t) \quad (7)$$

where M, C, and K = $n \times n$ time-invariant mass, damping, and stiffness matrices, respectively; $U(t)$, $\dot{U}(t)$, and $\ddot{U}(t)$ = length n vectors of nodal displacements, velocities, and accelerations, respectively; P = length n load distribution vector; and $F(t)$ = an external, scalar loading function, which, in the case of random excitations, is modeled as a random process. Eq. (7) can be recast into the "state variable" form by defining the following length $2n$ "state vector":

$$Z(t) = \begin{bmatrix} U(t) \\ \dot{U}(t) \end{bmatrix}_{(2n \times 1)} \quad (8)$$

The matrix equation of motion in (7) can be recast into the following first-order matrix equation:

$$\dot{Z}(t) = GZ(t) + \tilde{P}F(t) \quad (9)$$

where

$$G = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ (-M^{-1}K) & (-M^{-1}C) \end{bmatrix}_{(2n \times 2n)} \quad (10)$$

and

$$\tilde{P} = \begin{bmatrix} 0_{n \times 1} \\ M^{-1}P \end{bmatrix}_{(2n \times 1)} \quad (11)$$

COMPLEX MODAL ANALYSIS

The complex modal matrix T formed from the complex eigenmodes can be used as an appropriate transformation matrix

to decouple the first-order matrix equation in (9). Introducing the transformed state vector $V(t)$ of complex modal coordinates, where

$$Z(t) = TV(t) \quad (12)$$

and substituting into (9), one finds that

$$T\dot{V}(t) = GTV(t) + \tilde{P}F(t) \quad (13)$$

Premultiplying the foregoing equation by the inverse of the complex modal matrix T^{-1} gives

$$\dot{V}(t) = T^{-1}GTV(t) + T^{-1}\tilde{P}F(t) \quad (14)$$

Eq. (14) can be simplified by using the fact that the complex eigenvectors are orthogonal with respect to G (Reid 1983). Therefore

$$T^{-1}GT = D \quad (15)$$

where D is the diagonal matrix containing the $2n$ complex eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ of the system matrix G. Thus (14) can be rewritten as

$$\dot{V}(t) = DV(t) + T^{-1}\tilde{P}F(t) \quad (16)$$

and the complex modal coordinates $V_i(t)$ satisfy the following set of uncoupled first-order differential equations:

$$\dot{V}_i(t) = \lambda_i V_i(t) + \Gamma_i F(t), \quad i = 1, 2, \ldots, 2n \quad (17)$$

where Γ_i = ith modal participation factor (complex-valued) defined as the ith component of the vector $T^{-1}\tilde{P}$. Introducing the normalized complex modal response $S_i(t)$ as

$$V_i(t) = \Gamma_i S_i(t), \quad i = 1, 2, \ldots, 2n \quad (18)$$

and substituting into (17), one obtains the normalized complex modal equations

$$\dot{S}_i(t) = \lambda_i S_i(t) + F(t), \quad i = 1, 2, \ldots, 2n \quad (19)$$

The impulse response function for the ith mode $h_i(t)$ defined as the solution of (19) when $F(t) = \delta(t)$ where $\delta(t)$ denotes the Dirac delta function and for at rest initial conditions at time $t = 0^-$, is simply given by

$$h_i(t) = e^{\lambda_i t}, \quad t > 0 \quad (20)$$

Assuming for simplicity that the system is initially at rest, the solution of (19) can be expressed as the Duhamel integral

$$S_i(t) = \int_0^t e^{\lambda_i(t-\tau)}F(\tau)d\tau, \quad i = 1, 2, \ldots, 2n \quad (21)$$

It is worth mentioning that the normalized complex modal responses $S_i(t)$, $i = 1, 2, \ldots, 2n$, are complex conjugate by pairs. Combining (12) and (18) yields

$$Z(t) = TV(t) = T\Gamma S(t) = \tilde{T}S(t) \quad (22)$$

where Γ = diagonal matrix containing the $2n$ modal participation factors Γ_i; $\tilde{T} = T\Gamma$ = effective modal participation matrix; and $S = [S_1(t), S_2(t), \ldots, S_{2n}(t)]^T$ is the normalized complex modal response vector.

STOCHASTIC RESPONSE OF LINEAR DYNAMIC SYSTEMS

With the use of the evolutionary process theory (Priestley 1987), the loading function $F(t)$ can be expressed in Fourier-Stieltjes integral form as

$$F(t) = \int_{-\infty}^{\infty} a_F(\omega, t)e^{j\omega t}dZ(\omega) \quad (23)$$

where $j = \sqrt{-1}$; $a_F(\omega, t)$ = a frequency-time modulating function; and $dZ(\omega)$ = a zero-mean orthogonal-increment process having the property

$$E[dZ^*(\omega_1)dZ(\omega_2)] = \Phi(\omega_1)\delta(\omega_1 - \omega_2)d\omega_1 d\omega_2 \qquad (24)$$

where the superscript * denotes the complex conjugate operator and $\Phi(\omega)$ = PSD function characterizing $dZ(\omega)$. The stochastic response of both classically and nonclassically damped linear dynamic systems subjected to the input process $F(t)$ is derived in the time-frequency domain by using state-space and complex modal analysis. Substituting (23) into (21) gives

$$S_i(t) = \int_0^t e^{\lambda_i(t-\tau)} \int_{-\infty}^{\infty} a_F(\omega, \tau)e^{j\omega\tau}dZ(\omega)d\tau = \int_{-\infty}^{\infty} m_i(\omega, t)e^{j\omega t}dZ(\omega) \qquad (25)$$

where $m_i(\omega, t)$ = time-frequency modulating function of the ith normalized complex modal response and can be expressed as

$$m_i(\omega, t) = e^{-j\omega t} \int_0^t e^{\lambda_i(t-\tau)}a_F(\omega, \tau)e^{j\omega\tau}d\tau \qquad (26)$$

Assuming that the pth time derivative of $S_i(t)$ exists (in the mean-square sense), then

$$\frac{d^p}{dt^p} S_i(t) = S_i^{(p)}(t) = \int_{-\infty}^{\infty} \tilde{m}_i^{(p)}(\omega, t)e^{j\omega t}dZ(\omega) \qquad (27)$$

where the time-frequency modulating function of $S_i^{(p)}(t)$ is given by

$$\tilde{m}_i^{(p)}(\omega, t) = e^{-j\omega t} \frac{\partial^p}{\partial t^p} [m_i(\omega, t)e^{j\omega t}] \qquad (28)$$

Hence, the second-order statistics of the time derivatives of various orders of the normalized complex modal responses can be derived as

$$R_{S_i^{(p)}S_j^{(q)}}(t, \tau) = E[(S_i^{(p)}(t))^*S_j^{(q)}(t + \tau)]$$

$$+ \int_{-\infty}^{\infty} [\tilde{m}_i^{(p)}(\omega, t)]^*\Phi(\omega)[\tilde{m}_i^{(q)}(\omega, t + \tau)]e^{j\omega\tau}d\omega \qquad (29)$$

If $\tau = 0$, (29) reduces to

$$R_{S_i^{(p)}S_j^{(q)}}(t, 0) = \int_{-\infty}^{\infty} [\tilde{m}_i^{(p)}(\omega, t)]^*\Phi(\omega)[\tilde{m}_j^{(q)}(\omega, t)]d\omega$$

$$= \int_{-\infty}^{\infty} \Phi_{S_i^{(p)}S_j^{(q)}}(\omega, t)d\omega \qquad (30)$$

in which

$$\Phi_{S_i^{(p)}S_j^{(q)}}(\omega, t) = [\tilde{m}_i^{(p)}(\omega, t)]^*\Phi(\omega)[\tilde{m}_j^{(q)}(\omega, t)] \qquad (31)$$

denotes the evolutionary cross-PSD function of the derivatives of order p and q of the ith and jth normalized complex modal responses, respectively. From (22), the second-order statistics of the time derivatives of order p and q of the response vector $\mathbf{Z}(t)$ can be derived easily as

$$\mathbf{R}_{\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}}(t, \tau) = E[[\mathbf{Z}^{(p)}(t)]^*[\mathbf{Z}^{(q)}(t + \tau)]^T] = \tilde{\mathbf{T}}^*\mathbf{R}_{S^{(p)}S^{(q)}}(t, \tau)\tilde{\mathbf{T}}^T \qquad (32)$$

in which the components of the complex cross-modal correlation matrix $\mathbf{R}_{S^{(p)}S^{(q)}}(t, \tau)$ are given by (29) and the superscript T denotes the matrix transposition operator. Similarly, the EPSD matrix of $\mathbf{Z}^{(p)}(t)$ and $\mathbf{Z}^{(q)}(t)$ takes the form

$$\mathbf{\Phi}_{\mathbf{Z}^{(p)}\mathbf{Z}^{(q)}}(\omega, t) = \tilde{\mathbf{T}}^*\mathbf{\Phi}_{S^{(p)}S^{(q)}}(\omega, t)\tilde{\mathbf{T}}^T \qquad (33)$$

in which the components of the evolutionary complex cross-modal PSD matrix $\mathbf{\Phi}_{S^{(p)}S^{(q)}}(\omega, t)$ are given by (31).

EXPLICIT CLOSED-FORM SOLUTIONS FOR RESPONSE OF LINEAR MDOF SYSTEMS SUBJECTED TO NONSTATIONARY GROUND MOTION MODEL

The second-order statistics of the response of a linear MDOF system subjected to the fully nonstationary earthquake ground motion model presented earlier is derived using state-space and complex modal analysis. Because of the assumption in the ground motion model that the component processes of the sigma-oscillatory process are pairwise statistically independent, the second-order response statistics can be obtained by simply adding the contributions arising from the individual component processes.

Substituting the time-frequency modulating function $a_F(\omega, t)$ in (26) with the time modulating function $A_k(t)$ of the kth earthquake component process defined in (2), one finds the time-frequency modulating function of the ith complex normalized modal response to the kth earthquake component process as

$$m_i(\omega, t) = e^{-j\omega t} \int_0^t e^{\lambda_i(t-\tau)}\alpha_k\tau^{\beta_k}e^{-\gamma_k\tau}e^{j\omega\tau}d\tau$$

$$= \alpha_k\beta_k! \left[e^{-\gamma_k t} \sum_{n=0}^{\beta_k} \frac{(-1)^n t^{(\beta_k-n)}}{(\beta_k - n)!a^{n+1}} - \frac{(-1)^{\beta_k}e^{(\lambda_i-j\omega)t}}{a^{\beta_k+1}} \right] \qquad (34)$$

where $a = j\omega - \lambda_i - \gamma_k$. If $a = 0$, then

$$m_i(\omega, t) = \frac{\alpha_k e^{(\lambda_i-j\omega)t}t^{\beta_k+1}}{\beta_k + 1} \qquad (35)$$

It should be emphasized that in (34)–(41), t is defined as the relative time measured from ζ_k, the arrival time of the kth component process (see Appendix I). In the special case for which $\alpha_k = 1$ and $\beta_k = \gamma_k = 0$, the modulating function of the kth earthquake component process reduces to the unit-step function. Hence

$$m_i(\omega, t) = \frac{1 - e^{(\lambda_i-j\omega)t}}{(j\omega - \lambda_i)} \qquad (36)$$

For a stable linear system, the real parts of the eigenvalues are negative. Therefore, as $t \to \infty$, $m_i(\omega, t)$ in (36) reduces to

$$m_i(\omega, t) = \frac{1}{(j\omega - \lambda_i)} = h_i(\omega) \qquad (37)$$

which is the Fourier transform of the complex modal impulse response function as defined in (20). Thus, for a unit-step modulating function, the response becomes stationary as $t \to \infty$.

In the pure time domain, the cross-correlation function of complex normalized modal responses $S_i(t)$ and $S_j(t)$ due to the kth earthquake component process $X_k(t)$ can be derived as follows:

$$R_{S_iS_j}(t, \tau) = E[S_i^*(t)S_j(t + \tau)]$$

$$= \int_0^t e^{\lambda_i^*(t-u)} \int_0^{t+\tau} e^{\lambda_j(t+\tau-s)}E[X_k^*(u)X_k(s)]dsdu \qquad (38)$$

in which $\tau \geq 0$. For $\tau < 0$, the following relationships can be used:

$$R_{S_iS_j}(t, \tau) = R_{S_jS_i}^*(t + \tau, -\tau) \quad \text{if } t + \tau \geq 0$$

$$R_{S_iS_j}(t, \tau) = 0 \qquad\qquad\quad \text{if } t + \tau < 0 \qquad (39)$$

After extensive algebraic manipulations, the following explicit closed-form solution is obtained for the complex cross-modal cross-correlation function:

$$R_{S_iS_j}(t, \tau) = G_1(G_2 + G_3 + G_{41}G_{42} + G_{51}G_{52}) \qquad (40)$$

where

$$G_1 = \alpha_k^2 e^{(\lambda_i^* + \lambda_j)t + \lambda_j \tau}; \quad G_{41} = e^{a_2 t + a_4 \tau}(\beta_k!)^2; \quad G_{51} = e^{a_4(t+\tau)}(\beta_k!)^2$$

$$G_2 = \sum_{n=0}^{\beta_k} \frac{(-1)^{\beta_k - n}\beta_k![A_1(n) - A_4(n)]}{n!} \int_0^t e^{a_2 u} u^{\beta_k + n} du$$

$$= \beta_k! \sum_{n=0}^{\beta_k} \frac{(\beta_k + n)![A_1(n) - A_4(n)]}{n! a_2^{\beta_k + n + 1}} \left[e^{a_2 t} \sum_{m=0}^{\beta_k + n} \frac{(a_2 t)^m}{(-1)^m m!} - 1 \right]$$

$$G_3 = (\beta_k!)^2 e^{a_3 t} \sum_{n=0}^{\beta_k} \frac{(-1)^{2\beta_k - n} t^n}{n!} \{[B_1(0)A_2(n) - A_1(0)B_3(n)]\sin(\eta_k t)$$

$$+ [B_1(0)B_2(n) - A_1(0)A_3(n)]\cos(\eta_k t)\}$$

$$+ (\beta_k)^2[A_1(0)A_3(0) - B_1(0)B_2(0)]$$

$$G_{42} = \sum_{n=0}^{\beta_k} \frac{(t + \tau)^n}{n!} \sum_{m=0}^{\beta_k} \frac{t^m}{m!(-1)^{n+m}} \{[A_4(n)A_5(m)$$

$$- B_4(n)B_6(m)]\cos(\eta_k \tau) + [B_4(n)A_6(m) - A_4(n)B_5(m)]\sin(\eta_k \tau)\}$$

$$G_{52} = \sum_{n=0}^{\beta_k} \frac{(t + \tau)^n}{(-1)^n n!} \{[B_4(n)B_6(0) - A_4(n)A_5(0)]\cos[\eta_k(t + \tau)]$$

$$+ [A_4(n)B_5(0) - B_4(n)A_6(0)]\sin[\eta_k(t + \tau)]\}$$

$$a_1 = \nu_k - \lambda_j - \gamma_k; \quad a_2 = -(\lambda_i^* + \lambda_j + 2\gamma_k); \quad a_3 = -(\nu_k + \lambda_i^* + \gamma_k);$$

$$a_4 = -(\nu_k + \lambda_j + \gamma_k); \quad a_5 = \nu_k - \lambda_i^* - \gamma_k; \quad b = \eta_k$$

$$\begin{bmatrix} A_i(n-1) \\ B_i(n-1) \end{bmatrix} = \begin{bmatrix} A_i(\beta_k) & -B_i(\beta_k) \\ B_i(\beta_k) & A_i(\beta_k) \end{bmatrix} \begin{bmatrix} A_i(n) \\ B_i(n) \end{bmatrix};$$

$$n = 0, 1, \ldots, \beta_k; \quad i = 1, 2, \ldots, 6$$

$$A_1(\beta_k) = a_1/(a_1^2 + b^2); \quad B_1(\beta_k) = b/(a_1^2 + b^2)$$

$$A_2(\beta_k) = a_3/(a_3^2 + b^2); \quad B_2(\beta_k) = -b/(a_3^2 + b^2)$$

$$A_3(\beta_k) = a_3/(a_3^2 + b^2); \quad B_3(\beta_k) = b/(a_3^2 + b^2)$$

$$A_4(\beta_k) = a_4/(a_4^2 + b^2); \quad B_4(\beta_k) = b/(a_4^2 + b^2)$$

$$A_5(\beta_k) = a_5/(a_5^2 + b^2); \quad B_5(\beta_k) = b/(a_5^2 + b^2)$$

$$A_6(\beta_k) = a_5/(a_5^2 + b^2); \quad B_6(\beta_k) = -b/(a_5^2 + b^2)$$

For the special case when $\beta_k = \gamma_k = 0$ and $t \to \infty$, as mentioned earlier, the response becomes stationary with the cross-modal correlation function depending only on the time lag τ. The following stationary cross-modal cross-correlation function is obtained:

$$R_{S_iS_j}(t, \tau) = R_{S_iS_j}(\tau) = \alpha_k^2 e^{\lambda_j \tau} \left\{ \frac{A_1(\beta_k) - A_4(\beta_k)}{a_2} \right.$$

$$+ e^{a_4 \tau}\{[A_4(\beta_k)A_5(\beta_k) - B_4(\beta_k)B_6(\beta_k)]\cos(\eta_k \tau)$$

$$\left. + [B_4(\beta_k)A_6(\beta_k) - A_4(\beta_k)B_5(\beta_k)]\sin(\eta_k \tau)\} \right\} \qquad (41)$$

The second-order statistics of the nodal relative displacement and velocity responses of a linear MDOF system can be obtained from the second-order statistics of the normalized complex modal responses, simply by summing over all modes and over all subprocesses of the ground motion model accounting for their different arrival times.

If needed, the second-order statistics of the nodal absolute acceleration responses also can be derived through a simple linear transformation as follows. Consider the following alternative form of the governing equation of motion for the MDOF system:

$$\mathbf{M\ddot{X}}(t) + \mathbf{C\dot{U}}(t) + \mathbf{KU}(t) = \mathbf{0}_{n \times 1} \qquad (42)$$

where $\mathbf{\ddot{X}}(t)$ = length n absolute acceleration response vector. Thus

$$\mathbf{\ddot{X}}(t) = [(-\mathbf{M}^{-1}\mathbf{K}) \quad (-\mathbf{M}^{-1}\mathbf{C})] \begin{bmatrix} \mathbf{U}(t) \\ \mathbf{\dot{U}}(t) \end{bmatrix} = \mathbf{AZ}(t) \qquad (43)$$

where the matrix $\mathbf{A}$ = an $n \times 2n$ transformation matrix defined as

$$\mathbf{A} = [(-\mathbf{M}^{-1}\mathbf{K}) \quad (-\mathbf{M}^{-1}\mathbf{C})] \qquad (44)$$

Therefore, the correlation matrix and EPSD matrix of the absolute acceleration response vector $\mathbf{\ddot{X}}(t)$ are given by

$$\mathbf{R}_{\ddot{X}\ddot{X}}(t, \tau) = \mathbf{AR}_{ZZ}(t, \tau)\mathbf{A}^T \qquad (45)$$

$$\mathbf{\Phi}_{\ddot{X}\ddot{X}}(\omega, t) = \mathbf{A\Phi}_{ZZ}(\omega, t)\mathbf{A}^T \qquad (46)$$

CROSS-MODAL CORRELATION COEFFICIENTS FOR CLASSICAL MODAL RESPONSES TO NONSTATIONARY GROUND MOTION MODEL

The equations of a classically damped linear MDOF system can be decoupled into classical second-order modal equations such as

$$\ddot{\bar{S}}_i + 2\xi_i \omega_i \dot{\bar{S}}_i(t) + \omega_i^2 \bar{S}_i(t) = -\ddot{U}_g(t) \qquad (47a)$$

$$\ddot{\bar{S}}_j(t) + 2\xi_j \omega_j \dot{\bar{S}}_j(t) + \omega_j^2 \bar{S}_j(t) = -\ddot{U}_g(t) \qquad (47b)$$

where $\bar{S}_i(t)$ and $\bar{S}_j(t)$ = ith and jth normalized classical modal responses, respectively; and ξ_i and ω_i = ith modal damping ratio and undamped natural circular frequency, respectively. Using the state space approach, (47) can be recast into the first-order form

$$\begin{bmatrix} \dot{\bar{S}}_i(t) \\ \dot{\bar{S}}_j(t) \\ \ddot{\bar{S}}_i(t) \\ \ddot{\bar{S}}_j(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_i^2 & 0 & -2\xi_i \omega_i & 0 \\ 0 & -\omega_j^2 & 0 & -2\xi_j \omega_j \end{bmatrix} \begin{bmatrix} \bar{S}_i(t) \\ \bar{S}_j(t) \\ \dot{\bar{S}}_i(t) \\ \dot{\bar{S}}_j(t) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} \ddot{U}_g(t)$$

$$(48)$$

It can be shown that the eigenvalues $\{\lambda_i, i = 1, 2, 3, 4\}$ of the system matrix in (48) are given by $\{-\xi_i \omega_i \pm j\omega_{D_i}, -\xi_j \omega_{D_j} \pm j\omega_{D_j}\}$ in which $\omega_{D_i} = \omega_i \sqrt{1 - \xi_i^2}$ is the ith modal damped natural circular frequency. The eigenvectors associated with these eigenvalues are combined to form the following complex modal matrix:

$$\mathbf{T} = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{t}_3 \quad \mathbf{t}_4] = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \end{bmatrix} \qquad (49)$$

The importance of modal correlation is quantified through the following cross-modal correlation coefficients (Der Kiureghian 1980; Conte and Peng 1996):

$$\rho_{0,ij}(t) = \frac{R_{\bar{S}_i\bar{S}_j}(t, 0)}{\sqrt{R_{\bar{S}_i\bar{S}_i}(t, 0)R_{\bar{S}_j\bar{S}_j}(t, 0)}} \qquad (50)$$

$$\rho_{2,ij}(t) = \frac{R_{\dot{\bar{S}}_i\dot{\bar{S}}_j}(t, 0)}{\sqrt{R_{\dot{\bar{S}}_i\dot{\bar{S}}_i}(t, 0)R_{\dot{\bar{S}}_j\dot{\bar{S}}_j}(t, 0)}} \qquad (51)$$

$$\rho_{a,ij}(t) = \frac{R_{\ddot{S}_i^a\ddot{S}_j^a}(t, 0)}{\sqrt{R_{\ddot{S}_i^a\ddot{S}_i^a}(t, 0)R_{\ddot{S}_j^a\ddot{S}_j^a}(t, 0)}} \qquad (52)$$

The coefficients $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$, and $\rho_{a,ij}(t)$ can be interpreted physically as the correlation coefficients of the normalized classical modal displacement responses $\bar{S}_i(t)$ and $\bar{S}_j(t)$, of the normalized classical modal velocity responses $\dot{\bar{S}}_i(t)$ and $\dot{\bar{S}}_j(t)$ and of the normalized classical modal absolute acceleration responses $\bar{S}_i^a(t) = -2\xi_i\omega_i\dot{\bar{S}}_i(t) - \omega_i^2\bar{S}_i(t)$ and $\bar{S}_j^a(t)$, respectively.

Following the first-order complex modal decomposition method, the explicit closed-form solutions for the correlation coefficient functions $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$, and $\rho_{a,ij}(t)$ can be derived easily using (32) and (40). Similarly, the evolutionary cross-modal (classical modes) PSD functions can be found by making use of (31) and (33).

EFFECTS OF STATISTICAL CORRELATION OF CLASSICAL MODAL RESPONSES TO GROUND MOTION MODEL

The cross-modal correlation coefficients obtained for the ground motion model calibrated to the El Centro 1940 and Orion Blvd. 1971 earthquake records are displayed in Figs. 1–4. Figs. 1(a), 2(a), and 3(a) show the influence of the modal damping ratio on the cross-modal correlation coefficients for the El Centro 1940 earthquake ground motion model. The modal damping ratios $\xi_i = \xi_j = \xi$ vary from 2 to 10% of critical. The reference modal frequency ω_i is 20 rad/s, which is approximately the fundamental undamped natural circular frequency ($\omega_i \approx 2\pi/0.3$) of a three-story uniform building. The modal responses are considered five seconds after the beginning of the El Centro earthquake. The mean-squared modal displacement responses reach their maxima at approximately this time. These figures indicate that the importance of cross-modal correlation in computing mean-squared structural re-
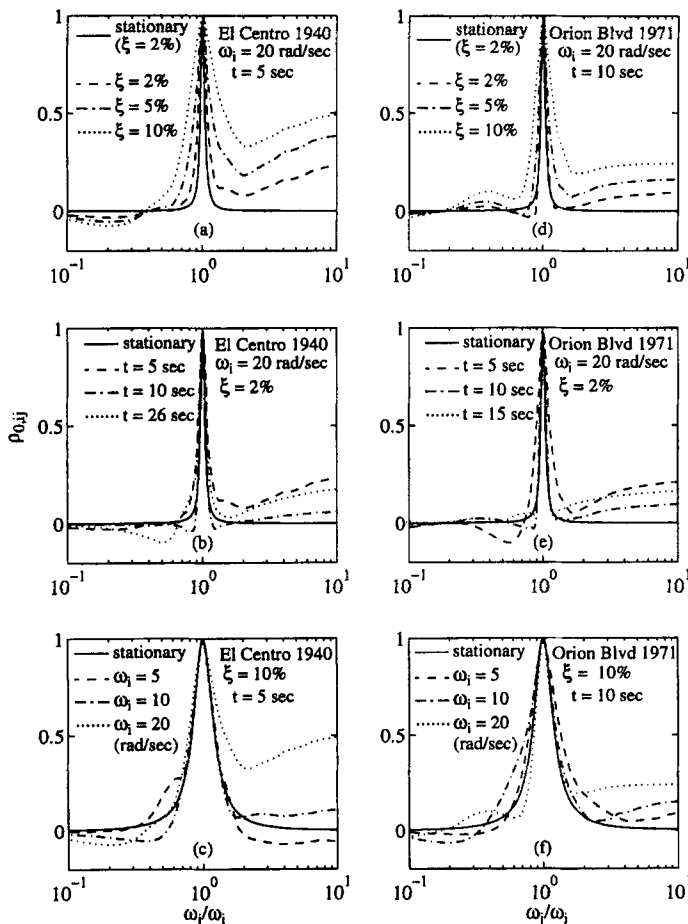

FIG. 1. Cross-Modal Correlation Coefficient $\rho_{0,ij}$ for Nonstationary Response to Earthquake Ground Motion Model
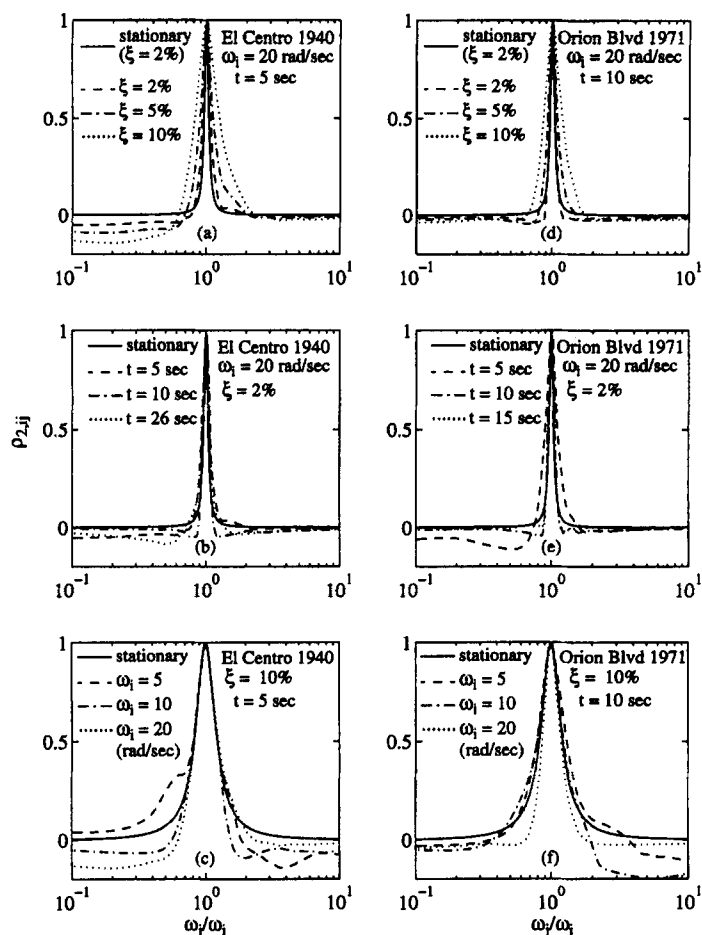
FIG. 2. Cross-Modal Correlation Coefficient $\rho_{2,ij}$ for Nonstationary Response to Earthquake Ground Motion Model

sponses increases with the modal damping ratio ξ. It is observed that the nonstationary cross-modal correlation coefficients obtained here are larger in absolute value than those for stationary response to white-noise input (Der Kiureghian 1980). Also, it is noticed that the nonstationary cross-modal correlation coefficients can be negative, do not necessarily decay to zero, and can increase as the cross-modal frequency ratio ω_j/ω_i departs from unity. As shown by Figs. 1(d), 2(d), and 3(d), the same trends also are found in the case of the Orion Blvd. 1971 earthquake ground motion model.

Next, Figs. 1(b), 2(b), and 3(b) display the time dependence of the cross-modal correlation coefficients in the case of the El Centro 1940 earthquake ground motion model. The reference modal frequency ω_i and the modal damping ratio ξ are taken as 20 rad/s and 2%, respectively. The times selected correspond to various local maxima of the mean-squared modal responses. It was found that the nonstationary cross-modal correlation coefficients are largest at a time corresponding approximately to the first local maximum of the mean-squared modal responses. Also notice that in the neighborhood of $\omega_j/\omega_i = 1$, the nonstationary cross-modal correlation coefficients are larger than those for stationary response to white-noise excitation. The previous remarks also apply in the case of the Orion Blvd. 1971 earthquake ground motion model.

Finally, the dependence of the nonstationary cross-modal correlation coefficients on the reference modal frequency ω_i is displayed in subplots (c) and (f) of Figs. 1–3. Unlike the case of stationary response to white-noise excitation, the nonstationary cross-modal correlation coefficients derived here depend on both ω_i and ω_j and not only on the cross-modal frequency ratio ω_j/ω_i.
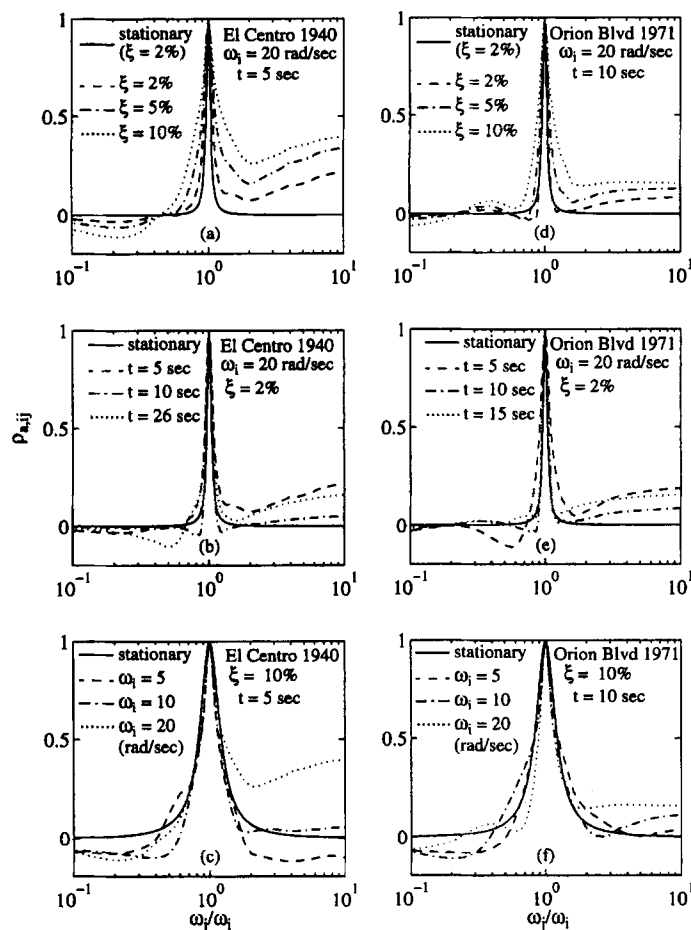
FIG. 3. Cross-Modal Correlation Coefficient $\rho_{a,ij}$ for Nonstationary Response to Earthquake Ground Motion Model

FIG. 4. Three-Dimensional Three-Story Unsymmetrical Building Example

APPLICATION EXAMPLE

To illustrate the application of complex modal analysis and the derived closed-form solutions for the response of linear MDOF systems subjected to the fully nonstationary earthquake ground motion model, an idealized three-dimensional unsymmetrical building is considered as shown in Fig. 4. This building consists of three floor diaphragms, assumed infinitely rigid in their own plane, supported by wide flange steel columns of size W14×145. Each floor diaphragm is assumed to be made of reinforced concrete with a weight density of 3.6 kN/m³ and a depth of 18 cm. The axial deformations of the columns are neglected. The modulus of elasticity of steel is 200 GPa. The motion of each floor diaphragm is defined completely by three DOFs defined at its center of mass (CM), namely, the relative displacements with respect to the ground in the x-direction $U_{X_i}(t)$, in the y-direction $U_{Y_i}(t)$, and the rotation about the vertical z-axis $\theta_{Z_i}(t)$. The earthquake ground motion excitation is assumed to act at 45° with respect to the x-axis. Both classically and nonclassically damped structural models are considered. For the case of classical damping, each modal damping ratio is taken as 2%. To physically realize the nonclassical damping case, diagonal viscous damping elements (fluid viscous braces) are added as shown in Fig. 4. The damping coefficient of each viscous damping element is taken as 0.1 kN·s per mm. The undamped natural circular frequencies of this building are shown in Table 1.

Figs. 5–11 show some second-order response statistics for the classically damped case, whereas Figs. 12 and 13 correspond to the nonclassical damping case. Fig. 5 gives the autocorrelation function of $U_{X_3}(t)$, the relative displacement in the x-direction of the third floor's CM, due to the El Centro
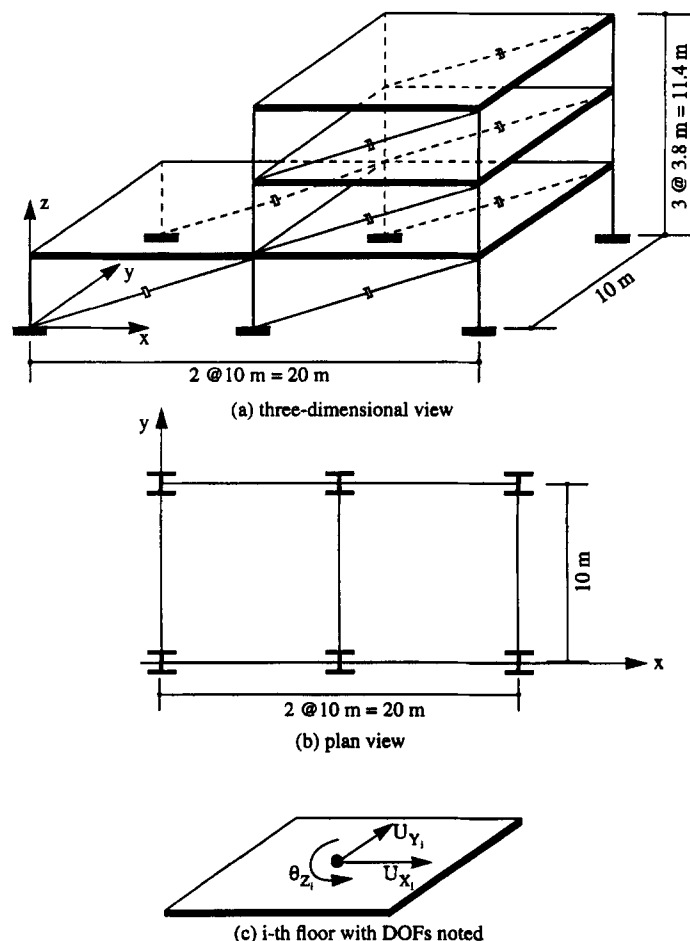
TABLE 1. Undamped Natural Frequencies and Description of Mode Shapes of Three-Dimensional Unsymmetrical Building Example

Mode number (1)	ω_i (rad/s) (2)	Description of mode shapes (3)
1	15.97	Translation in x-direction
2	24.12	Lateral-torsional coupling in y-direction
3	36.56	Translation in x-direction
4	41.21	Lateral-torsional coupling in y-direction
5	56.74	Lateral-torsional coupling in y-direction
6	56.98	Translation in x-direction
7	73.88	Lateral-torsional coupling in y-direction
8	95.15	Lateral-torsional coupling in y-direction
9	127.69	Lateral-torsional coupling in y-direction

1940 earthquake ground motion model. The autocorrelation function exhibits oscillatory decay along the time lag axis and behaves like the mean-square function of the El Centro 1940 earthquake ground acceleration model. At zero time lag ($\tau = 0$), the autocorrelation function reduces to the mean-squared response of $U_{X_3}(t)$. Fig. 6 displays the autocorrelation function of the rotational response $\theta_{Z_3}(t)$. Although the two autocorrelation functions represented in Figs. 5 and 6 are caused by the same stochastic ground motion model, their behaviors are quite different, especially along the time lag axis. The frequency of the oscillatory decay along the time lag axis in Fig. 6 is obviously lower than that in Fig. 5. Figs. 7 and 8 represent the autocorrelation function of the angular velocity response $\dot{\theta}_{Z_3}(t)$ and the cross-correlation function of the responses $\dot{\theta}_{Z_1}(t)$ and $\dot{\theta}_{Z_3}(t)$ to the Orion Blvd. 1971 earthquake ground motion model. Based on Figs. 5–8, it is worth noting that the
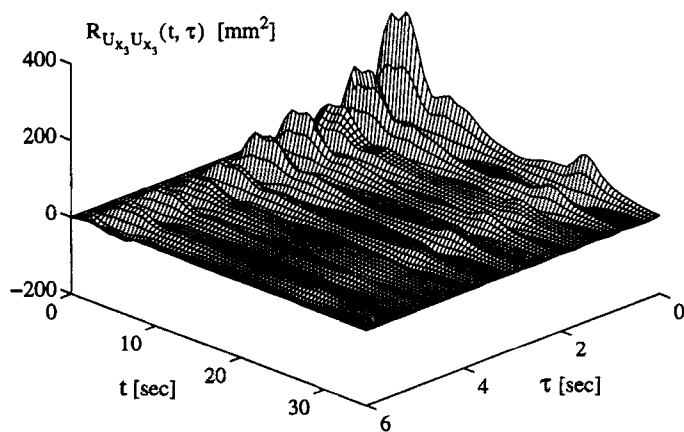
FIG. 5. Autocorrelation Function of Relative Displacement Response $U_{x_3}(t)$ to El Centro 1940 Earthquake Ground Motion Model
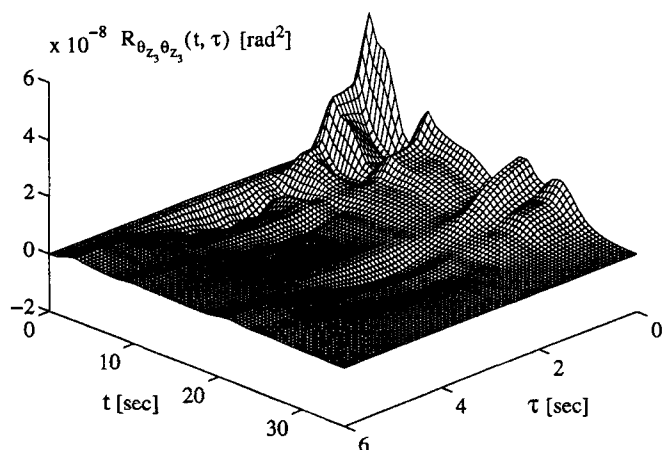

FIG. 6. Autocorrelation Function of Rotational Response $\theta_{z_3}(t)$ to El Centro 1940 Earthquake Ground Motion Model
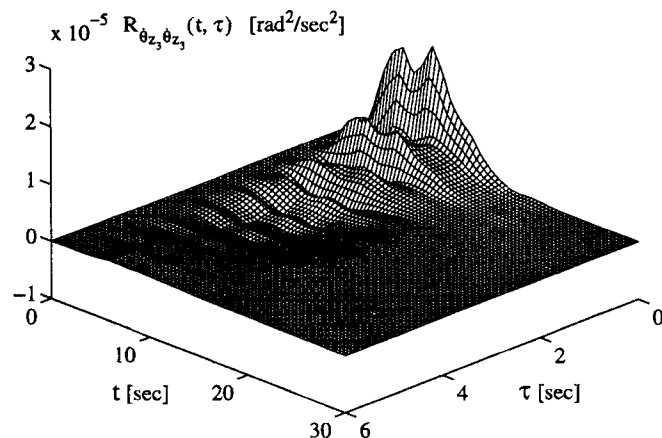

FIG. 7. Autocorrelation Function of Angular Velocity Response $\dot{\theta}_{z_3}(t)$ to Orion Blvd. 1971 Earthquake Ground Motion Model

rate of decay along the time lag axis of the correlation functions for rotational responses is larger than that for translational responses.

Next, EPSD functions corresponding to various response quantities are plotted in Figs. 9–13. The results given in Figs. 9 and 10 correspond to the El Centro 1940 earthquake ground motion model, while those in Figs. 11–13 have been obtained for the Orion Blvd. 1971 earthquake ground motion model. In Figs. 9–11, classical damping is assumed, while Figs. 12 and

13 correspond to the nonclassical damping case. Fig. 9 shows the EPSD function of the relative displacement response $U_{x_3}(t)$. From this figure, it is observed that the response $U_{x_3}(t)$ is mostly contributed by the first mode. This also is true for other translational displacement and velocity responses in the x-direction. Similarly, the translational displacement and velocity responses in the y-direction are almost entirely contributed by the second mode. As shown later, this is the reason why the mean-squared translational displacement and velocity responses can be estimated very accurately by neglecting the
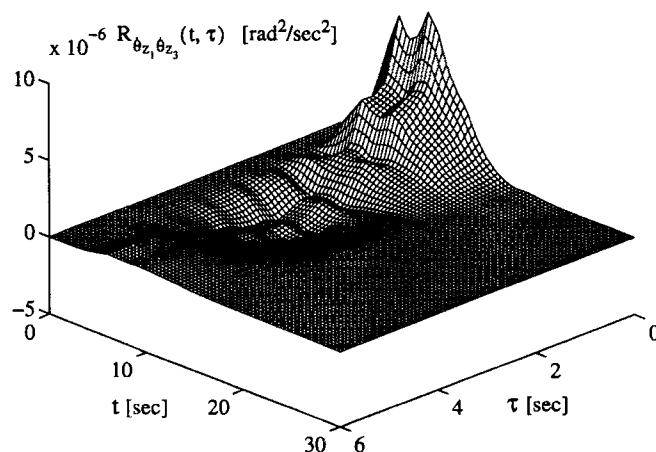

FIG. 8. Cross-Correlation Function of Angular Velocity Responses $\dot{\theta}_{z_1}(t)$ and $\dot{\theta}_{z_3}(t)$ to Orion Blvd. 1971 Earthquake Ground Motion Model
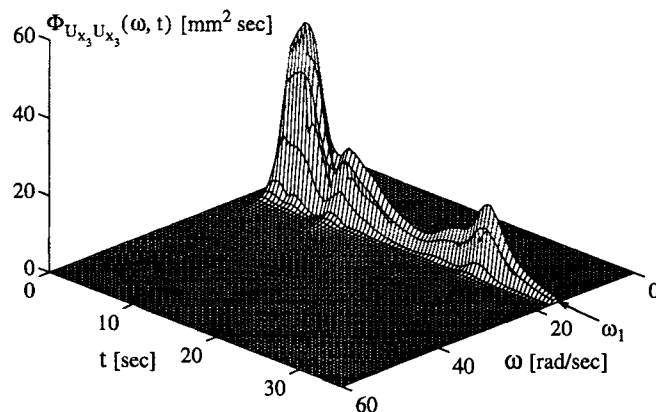

FIG. 9. Evolutionary Power Spectral Density Function of Relative Displacement Response $U_{x_3}(t)$ to El Centro 1940 Earthquake Ground Motion Model
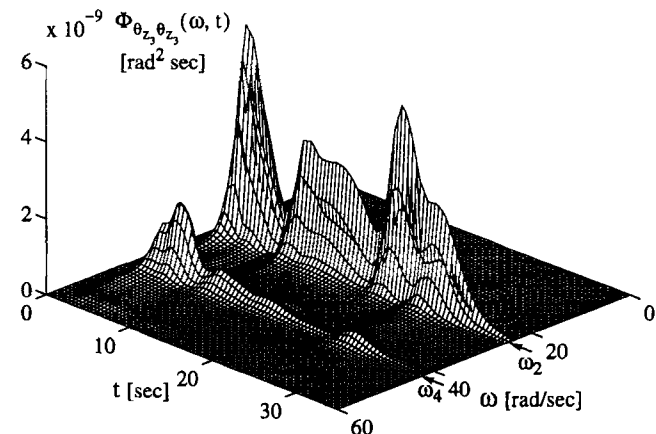

FIG. 10. Evolutionary Power Spectral Density Function of Rotational Response $\theta_{z_3}(t)$ to El Centro 1940 Earthquake Ground Motion Model

$\times 10^{-6}$ $\Phi_{\dot{\theta}_{z_3}\dot{\theta}_{z_3}}(\omega, t)$ [rad²/sec]

FIG. 11. Evolutionary Power Spectral Density Function of Angular Velocity Response $\dot{\theta}_{z_3}(t)$ to Orion Blvd. 1971 Earthquake Ground Motion Model

$\times 10^{-6}$ $\Phi_{\dot{\theta}_{z_3}\dot{\theta}_{z_3}}(\omega, t)$ [rad²/sec] (non-classical damping case)
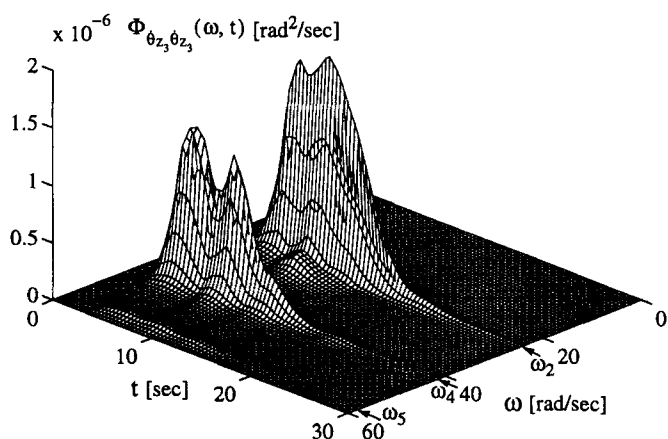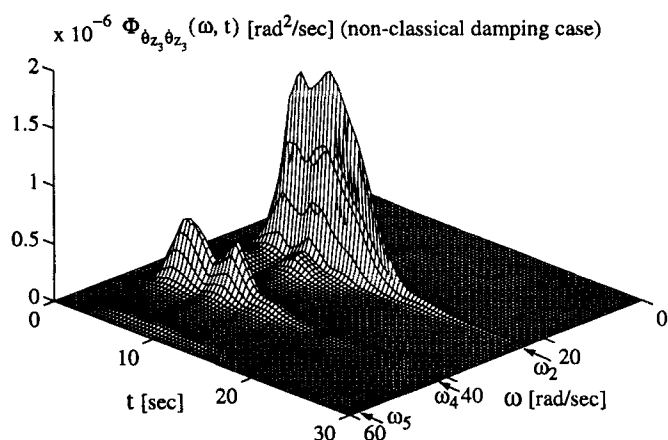
FIG. 12. Evolutionary Power Spectral Density Function of Angular Velocity Response $\dot{\theta}_{z_1}(t)$ to Orion Blvd. 1971 Earthquake Ground Motion Model

$\times 10^{-7}$ $\mathrm{Real}[\Phi_{\dot{\theta}_{z_3}\dot{\theta}_{z_1}}(\omega, t)]$ [rad²/sec] (non-classical damping case)

$\mathrm{Imag}[\Phi_{\dot{\theta}_{z_3}\dot{\theta}_{z_1}}(\omega, t)]$ [rad²/sec] (non-classical damping case)

$\times 10^{-8}$

FIG. 13. Evolutionary Cross-Power Spectral Density Function of the Angular Velocity Responses $\dot{\theta}_{z_3}(t)$ and $\dot{\theta}_{z_1}(t)$ to Orion Blvd. 1971 Earthquake Ground Motion Model

cross-modal correlations. Fig. 10 displays the EPSD function of the rotational response $\theta_{z_3}(t)$. The second and fourth modes contribute significantly to this response quantity because they are the two lowest modes containing rotational components. This also is true for both the EPSD function $\Phi_{\dot{\theta}_{z_3}\dot{\theta}_{z_3}}(\omega, t)$, of the angular velocity response $\dot{\theta}_{z_3}(t)$ (see Figs. 11 and 12) and the cross-PSD function $\Phi_{\dot{\theta}_{z_3}\dot{\theta}_{z_1}}(\omega, t)$, of angular velocity responses $\dot{\theta}_{z_3}(t)$ and $\dot{\theta}_{z_1}(t)$ (see Fig. 13). Note that this evolutionary cross-PSD function is complex-valued.

Figs. 14 and 15 present various mean-squared response quantities for the Orion Blvd. 1971 earthquake ground motion model and compare the exact results with those obtained by neglecting the cross-modal correlations. As already mentioned, the shape of the mean-squared responses is similar to that of the mean-squared earthquake ground acceleration process. The mean-squared rotational and angular velocity responses are overestimated if the cross-modal correlations are neglected [see subplots (c) and (f) of Figs. 14 and 15]. The translational displacement and velocity responses can be obtained accurately by neglecting the cross-modal correlations. This was already predicted earlier by the fact that the translational response in the x-direction is contributed by the first mode only (see Fig. 9), while two modes contribute significantly to the rotational response (see Figs. 10–13). Note that the mean-squared translational response in the x-direction is almost reduced by 50% as the viscous damping elements are added in that direction, while the mean-squared translational response in the y-direction remains almost unchanged. Similar results,
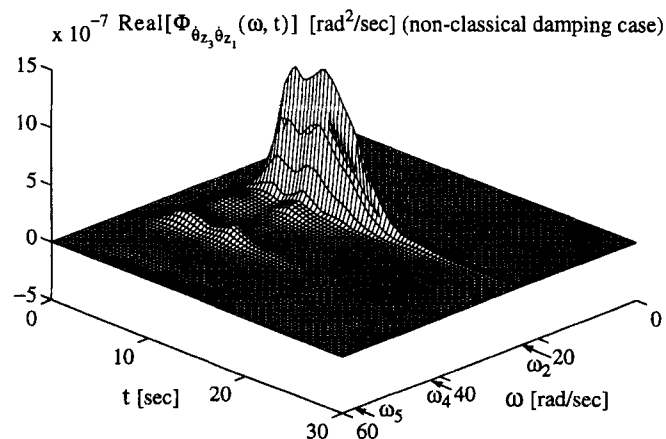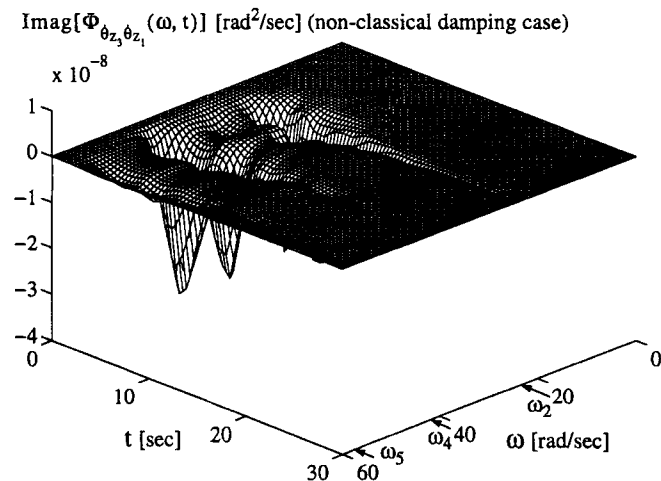
not shown here, also were obtained for the case of the El Centro 1940 earthquake ground motion model.

CONCLUSIONS

The nonstationary response of both classically and nonclassically damped linear MDOF systems to a newly developed fully nonstationary earthquake ground motion model is studied. A modal decomposition method is developed using the state-space approach and complex modal analysis. New explicit closed-form solutions are obtained for the evolutionary correlation and PSD matrices of the vector response process. Using these explicit closed-form solutions, the effects of statistical cross-modal correlations on the mean-squared structural response are investigated for the classical damping case. An application example is considered in which a three-dimensional unsymmetrical building with and without viscous bracing elements is subjected to the earthquake ground motion model acting at an angle with respect to the principal direction of the building. For the purpose of this study, the nonstationary earthquake ground motion model is calibrated against two actual, well-known earthquake ground acceleration records.

The nonstationary cross-modal correlation coefficient decays fast as the cross-modal frequency ratio ω_j/ω_i departs from one in a narrow band around one. However, unlike the case of stationary response to white-noise or filtered white-noise excitations, the nonstationary cross-modal correlation coefficients derived here can rise gradually as ω_j/ω_i departs further outside that region. The cross-modal correlation coefficients for the fully nonstationary earthquake ground motion model used in this study vary with time and depend on the modal
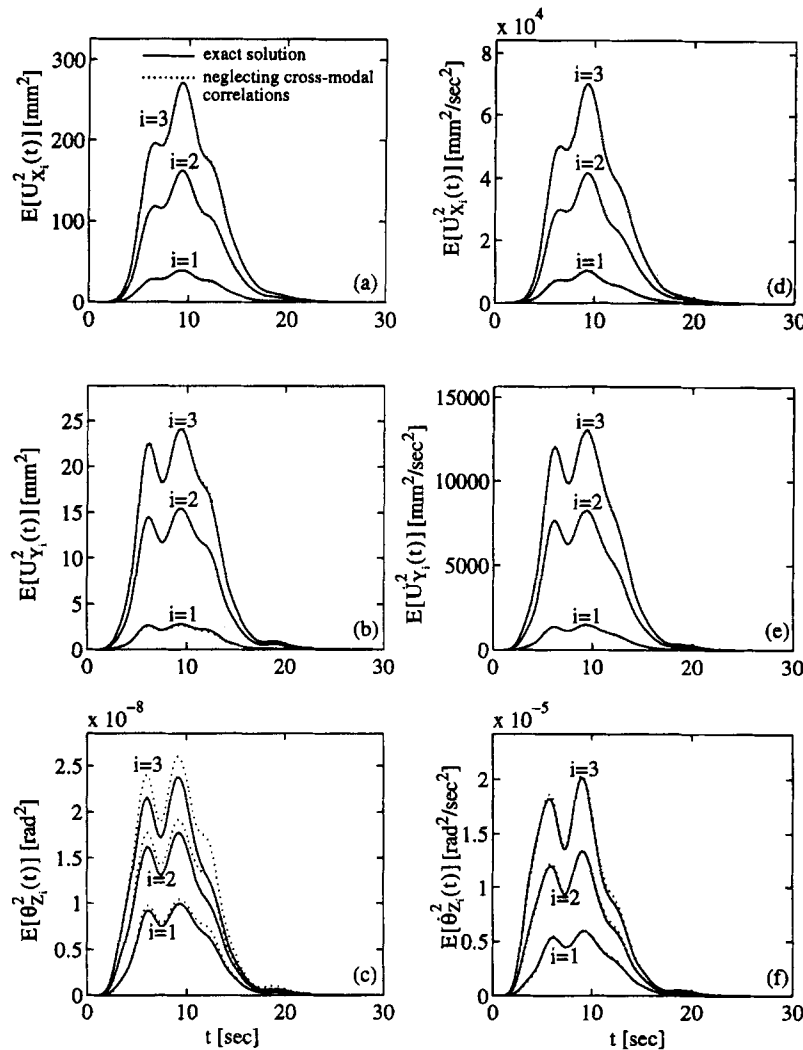
FIG. 14. Mean-Squared Responses to Orion Blvd. 1971 Earthquake Ground Motion Model (Classical Damping Case, Modal Damping Ratios = 2%)

damping ratios, the cross-modal frequency ratio ω_j/ω_i, and the reference modal frequency ω_i. In general, the nonstationary cross-modal correlation coefficients tend to be larger, in absolute value, than their counterparts for stationary response to white-noise excitations. It was found that the nonstationary cross-modal correlation coefficients can be negative. Future studies will compare the closed-form solutions obtained here for the time-varying cross-modal correlation coefficients and cross-modal PSD functions with the solutions obtained from other nonstationary earthquake ground motion models.

In the case of the three-dimensional unsymmetrical building considered, the mean-squared rotational responses of the rigid floor diaphragms could not be estimated accurately without accounting for cross-model correlations. This was because of the presence of relatively closely spaced modes with lateral-torsional coupling.

APPENDIX I. DERIVATION OF TIME-FREQUENCY MODULATING FUNCTION FOR Kth EARTHQUAKE COMPONENT PROCESS

The time-frequency modulating function $m_i(\omega, t)$ of the ith normalized complex modal response $S_i(t)$ is rewritten as follows:

$$m_i(\omega, t') = e^{-j\omega t'} \int_0^{t'} e^{\lambda_i(t'-\tau')} a_F(\omega, \tau') e^{j\omega\tau'} d\tau' \qquad (53)$$

where t' = absolute time variable, with respect to which the

arrival times $\zeta_1, \zeta_2, \ldots, \zeta_p$, of the earthquake component processes $X_1(t), X_2(t), \ldots, X_p(t)$ are measured. The time modulating function of the kth earthquake component process is rewritten from (2) as

$$a_F(\omega, \tau') = \alpha_k(\tau' - \zeta_k)^{\beta_k} e^{-\gamma_k(\tau'-\zeta_k)} H(\tau' - \zeta_k) \qquad (54)$$

Substituting (54) into (53) gives

$$m_i(\omega, t') = e^{-j\omega t'} \int_0^{t'} e^{\lambda_i(t'-\tau')} \alpha_k(\tau' - \zeta_k)^{\beta_k} e^{-\gamma_k(\tau'-\zeta_k)} H(\tau' - \zeta_k) e^{j\omega\tau'} d\tau'$$

$$= e^{-j\omega t'} \int_{\zeta_k}^{t'} e^{\lambda_i(t'-\tau')} \alpha_k(\tau' - \zeta_k)^{\beta_k} e^{-\gamma_k(\tau'-\zeta_k)} e^{j\omega\tau'} d\tau' \qquad (55)$$

By applying the change of variable $\tau = \tau' - \zeta_k$, the foregoing equation reduces to

$$m_i(\omega, t') = e^{-j\omega t'} \int_0^{(t'-\zeta_k)} e^{\lambda_i(t'-\zeta_k-\tau)} \alpha_k\tau^{\beta_k} e^{-\gamma_k\tau} e^{j\omega(\tau+\zeta_k)} d\tau$$

$$= e^{-j\omega(t'-\zeta_k)} \int_0^{(t'-\zeta_k)} e^{\lambda_i(t'-\zeta_k-\tau)} \alpha_k\tau^{\beta_k} e^{-\gamma_k\tau} e^{j\omega\tau} d\tau \qquad (56)$$

Introducing the relative time variable $t = t' - \zeta_k$, (56) becomes

$$m_i(\omega, t) = e^{-j\omega t} \int_0^t e^{\lambda_i(t-\tau)} \alpha_k\tau^{\beta_k} e^{-\gamma_k\tau} e^{j\omega\tau} d\tau \qquad (57)$$
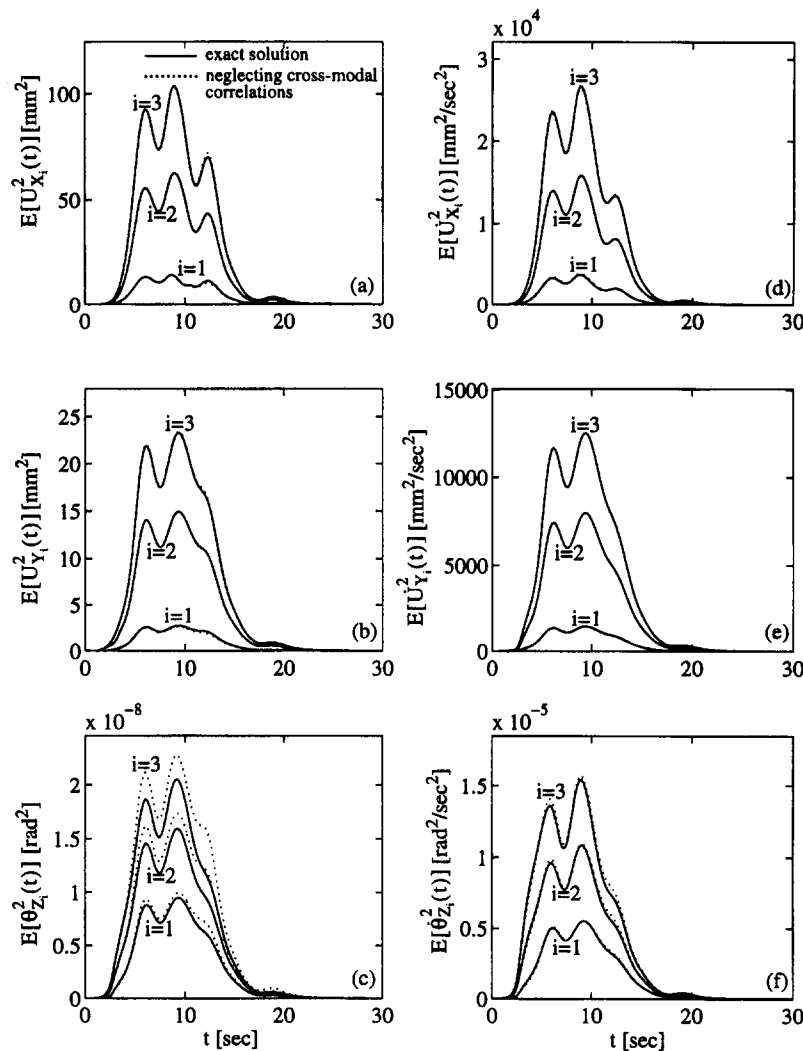
FIG. 15. Mean-Squared Responses to Orion Blvd. 1971 Earthquake Ground Motion Model (Nonclassical Damping Case)

ACKNOWLEDGMENTS

APPENDIX II. REFERENCES

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APPENDIX III.   NOTATION

The following symbols are used in this paper:

$A_k(t)$ = time modulating function of kth earthquake component process;

$\mathbf{C}$ = ($n \times n$) time-invariant damping matrix;

$\mathbf{D}$ = diagonal matrix of $2n$ eigenvalues of system matrix $\mathbf{G}$;

$dZ(\omega)$ = orthogonal-increment process;

$E[\ ]$ = expectation operator;

$F(t)$ = external loading process;

$\mathbf{G}$ = ($2n \times 2n$) time-invariant system matrix;

$H(t)$ = Heaviside unit step function;

$h_i(t)$ = ith modal impulse response function;

$h_i(\omega)$ = ith modal complex frequency response function;

$\mathbf{K}$ = ($n \times n$) time-invariant stiffness matrix;

$\mathbf{M}$ = ($n \times n$) time-invariant mass matrix;

$m_i(\omega, t)$ = time-frequency modulating function of $S_i(t)$;

$\tilde{m}_i^{(p)}(\omega, t)$ = time-frequency modulating function of $S_i^{(p)}(t)$;

$\mathbf{P}$ = length n load distribution vector;

$R_{S_k S_k}(\tau)$ = autocorrelation function of $S_k(t)$;

$R_{S_i^{(p)} S_j^{(q)}}(t, \tau)$ = cross-correlation function of $S_i^{(p)}(t)$ and $S_j^{(q)}(t)$;

$S_k(t)$ = stationary Gaussian process defining kth earthquake component process;

$S_i(t)$ = ith normalized modal displacement response;

$S_i^a(t)$ = ith normalized modal absolute acceleration response;

$S_i^{(p)}(t)$ = pth time derivative of $S_i(t)$;

$\mathbf{T}$ = $2n \times 2n$ complex modal matrix;

$\ddot{U}_g(t)$ = earthquake ground acceleration process;

$\mathbf{U}(t)$ = length n relative displacement response vector;

$\dot{\mathbf{U}}(t)$ = length n relative velocity response vector;

$\ddot{\mathbf{U}}(t)$ = length n relative acceleration response vector;

$\ddot{\mathbf{X}}(t)$ = length n absolute acceleration response vector;

$\mathbf{Z}(t)$ = length $2n$ state vector;

$\boldsymbol{\Gamma}$ = diagonal matrix of $2n$ modal participation factors;

$\delta(\ )$ = Dirac delta function;

λ_i = ith eigenvalue of system matrix $\mathbf{G}$;

ξ_i = ith modal damping ratio;

$\rho_{0,ij}(t)$ = correlation coefficient of $S_i(t)$ and $S_j(t)$;

$\rho_{2,ij}(t)$ = correlation coefficient of $\dot{S}_i(t)$ and $\dot{S}_j(t)$;

$\rho_{a,ij}(t)$ = correlation coefficient of $S_i^a(t)$ and $S_j^a(t)$;

$\Phi_{S_k S_k}(\omega)$ = PSD function of $S_k(t)$;

$\boldsymbol{\Phi}_{\ddot{X}\ddot{X}}(t, \omega)$ = EPSD matrix of $\ddot{\mathbf{X}}(t)$;

$\boldsymbol{\Phi}_{UU}(t, \omega)$ = EPSD matrix of $\mathbf{U}(t)$;

$\Phi_{S_i^{(p)} S_j^{(q)}}(\omega, t)$ = cross-PSD function of $S_i^{(p)}(t)$ and $S_j^{(q)}(t)$;

ω_i = ith modal undamped natural circular frequency; and

ω_{D_i} = ith modal damped natural circular frequency.