PROBABILISTIC ENGINEERING MECHANICS

Probabilistic Engineering Mechanics 14 (1999) 301–310

# On one-dimensional random fields with fixed end values

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Received 1 December 1997; received in revised form 1 July 1998; accepted 1 August 1998

**Abstract**

For a one-dimensional, uni-variate random field with deterministic fixed end values, expressions are derived for the conditional mean, variance, and covariance functions in terms of given mean, variance, and correlation functions for an unrestricted, variance-homogeneous Gaussian random field. Also, a relation is derived between the conditional random field and the underlying unrestricted random field. This relation is useful for simulation purposes. Further, expressions are derived for the coefficients in a series expansion for the conditional random field. The present results are obtained from known general formulas for conditional Gaussian distributions, conditional estimation, and series expansion. An earlier alternate approach to enforcing end conditions is also examined. An example is given to illustrate the effect of conditioning a random field by zero end constraints. The present results have direct application to the representation of random imperfections in probabilistic stability analysis of columns and arches. 

*Keywords:* Uni-variate conditional random field; Conditional estimation; Gaussian distribution

## 1. Introduction

If a one-dimensional (1D), uni-variate (1V) random field on (0,1) is required to take deterministic fixed values at 0 and 1, then the statistical properties of the resulting conditional random field depend on the fixed end values. Further, the conditional random field is necessarily nonhomogeneous. Here we derive expressions for the mean, variance, and covariance functions (first and second moment functions) of the conditional random field in terms of the known mean and correlation coefficient functions of the underlying unrestricted, variance-homogeneous,[2] Gaussian random field. Also, we derive a functional relationship between the unrestricted random field and the corresponding conditional random field. These results for a 1D-1V random field are obtained from general results based on the matrix formulas for conditional Gaussian distributions and conditional linear estimation (based on variance minimization) given by Vanmarcke [1].

Our general results agree with those obtained by Shinozuka and Zhang [2] using a different approach. In particular, we confirm their conclusion that the mean and covariance functions of the derived conditional random field agree with those obtained for the conditional Gaussian distribution. Thus, simulations based on the conditional distribution function and the conditional random field relation are equivalent under the Gaussian assumption, which is an essential equivalence for proper simulation [2].

An alternate method of imposing end constraints on a 1D-1V random field was proposed by Elishakoff [3]. For zero end conditions, his method consists of subtracting from the unrestricted random function its random end values times linear weight functions. Here we derive expressions for the mean, variance, and covariance functions for general weight functions from which Elishakoff's results [3] follow as a special case.

In addition, we derive formulas for the coefficients in a general series expansion for the conditional random field using our derived conditional mean and covariance functions in the formulation of Zhang and Ellingwood [4]. This generalizes previous results by Elishakoff [3] for the sine series expansion of a conditional random field based on his linear weight function approach.

The paper concludes with an example that illustrates the effect of conditioning a 1D-1V random field by requiring zero end values. Comparison is made between the results of the present conditional simulation method based on minimum-variance linear estimation and Elishakoff's linear weight function method [3]. For a random field with short correlation length, the variance of the conditional random

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[2] The case of inhomogeneous variance can also be treated, but the resulting formulas are rather cumbersome.

0266-8920/99/$ - see front matter 
PII: S0266-8920(98)00037-X

field by Elishakoff's method is much higher than that of the conditional random field by the variance minimization method. For large values of correlation length (on the order of the interval length), the results of the two methods differ by only a small amount. An example of a series expansion for a conditional random field also is given.

The results presented here have direct application to the characterization of random fields for modeling imperfections in columns and arches.

## 2. General results

Let $U(z)$ be a one-dimensional, uni-variate random field having mean function $\mu(z)$, variance function $\sigma^2(z)$, and covariance function $C(z,z')$, defined by

$$E[U(z)] = \mu(z) \; E[(U(z) - \mu(z))^2] = \sigma^2(z)$$
$$E[(U(z) - \mu(z))(U(z') - \mu(z'))] = C(z,z') \tag{1}$$

where $E[\cdots]$ denotes the mathematical expectation or ensemble average operator. Following the vector–matrix formulation of Vanmarcke [1], we consider the points $z_i$ ($i = 1,2,\ldots,n$) and introduce the vectors

$$\mathbf{U} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}, \quad \boldsymbol{\mu} = E[\mathbf{U}] = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix},$$

$$U_i = U(z_i), \;\; \mu_i = \mu(z_i) = E[U_i] \tag{2}$$

and covariance matrix

$$\mathbf{C} = E[(\mathbf{U} - \boldsymbol{\mu})(\mathbf{U}' - \boldsymbol{\mu}')] = \begin{bmatrix} \sigma_1^2 & C_{12} & \cdots & C_{1n} \\ C_{12} & \sigma_2^2 & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \ldots & \sigma_n^2 \end{bmatrix} \tag{3}$$

$$\sigma_i^2 = \sigma^2(z_i), \;\; C_{ij} = C(z_i, z_j)$$

where prime denotes the matrix transpose operation.

### 2.1. Conditional probability density function

In addition, let $U(z)$ be a Gaussian field so that the point-value variates $U_i = U(z_i)$, ($i = 1,\ldots,n$), are normally distributed with joint probability density function (PDF)

$$f_{U_1\ldots U_n}(u_1,\ldots,u_n) = \frac{1}{\sqrt{(2\pi)^n|\mathbf{C}|}}\exp\left[-\frac{1}{2}(\mathbf{u}' - \boldsymbol{\mu}')C^{-1}(\mathbf{u} - \boldsymbol{\mu})\right] \tag{4}$$

where the covariance matrix $\mathbf{C}$ is given by Eq. (3) and $|\mathbf{C}|$ denotes the determinant of $\mathbf{C}$. In order to find the conditional distribution when the first $k$ variates of $\mathbf{U}_i$ take prescribed values, following Vanmarcke [1], $\mathbf{U}$ is partitioned into two vectors $\mathbf{U}_1$ and $\mathbf{U}_2$ of dimension $k$ and $n - k$, respectively. The mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$ are similarly partitioned with the notation

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{12}' & \mathbf{C}_{22} \end{bmatrix}, \; \mathbf{U} = \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix}, \; \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}, \; \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix} \tag{5}$$

Then, as shown by Vanmarcke [1], the distribution of $\mathbf{U}_2$ given $\mathbf{U}_1$ is Gaussian with conditional PDF

$$f_{\mathbf{U}_2|\mathbf{U}_1}(\mathbf{u}_2|\mathbf{u}_1) = \frac{1}{\sqrt{(2\pi)^{n-k}|\mathbf{C}_{22|1}|}} \times \exp\left[-\frac{1}{2}(\mathbf{u}_2' - \boldsymbol{\mu}_{2|1}')\mathbf{C}_{22|1}^{-1}(\mathbf{u}_2 - \boldsymbol{\mu}_{2|1})\right] \tag{6}$$

where the conditional mean vector and covariance matrix are given by

$$\boldsymbol{\mu}_{2|1} = \boldsymbol{\mu}_2 + \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}(\mathbf{u}_1 - \boldsymbol{\mu}_1)$$
$$\mathbf{C}_{22|1} = \mathbf{C}_{22} - \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\mathbf{C}_{12} \tag{7}$$

The relations of Shinozuka and Zhang [2] for the conditional mean, covariance, and PDF can be obtained from Eqs. (4) and (7).

### 2.2. Conditional estimation

The optimal (minimum-variance) linear unbiased estimation for the conditional variate $\mathbf{U}_2$ given that $\mathbf{U}_1 = \bar{\mathbf{u}}_1$, denoted by $\mathbf{U}_{2|1}$, is shown by Vanmarcke [1] to be

$$\hat{\mathbf{u}}_2 = E[\mathbf{U}_{2|1}] = \boldsymbol{\mu}_{2|1} = \boldsymbol{\mu}_2 + \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}(\bar{\mathbf{u}}_1 - \boldsymbol{\mu}_1) \tag{8}$$

and the posterior (conditional) covariance of $\mathbf{U}_{2|1}$ is

$$\mathbf{C}_{22|1} = \mathbf{C}_{22} - \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\mathbf{C}_{12} \tag{9}$$

Thus, $\mathbf{U}_{2|1}$ can be written as

$$\mathbf{U}_{2|1} = \boldsymbol{\mu}_2 + \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}(\bar{\mathbf{u}}_1 - \boldsymbol{\mu}_1) + \mathbf{V}_2 \tag{10}$$

where $\mathbf{V}_2$ is a random variate that satisfies

$$E[\mathbf{V}_2] = 0, \;\; E\left[(\mathbf{U}_{2|1} - \hat{\mathbf{u}}_2)(\mathbf{U}_{2|1} - \hat{\mathbf{u}}_2)'\right] = E[\mathbf{V}_2, \mathbf{V}_2'] = \mathbf{C}_{22|1} \tag{11}$$

Using Eq. (9), it is not difficult to verify that a solution to Eq. (11) is

$$\mathbf{V}_2 = (\mathbf{U}_2 - \boldsymbol{\mu}_2) - \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}(\mathbf{U}_1 - \boldsymbol{\mu}_1) \tag{12}$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are variates which satisfy Eqs. (3) and (5), but need not be Gaussian. Then, from Eqs. (10) and (12), the conditional variate can be written as

$$\mathbf{U}_{2|1} = \hat{\mathbf{u}}_2^e + \mathbf{E}_2 \tag{13}$$

where

$$\bar{\mathbf{u}}_2^e = \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\bar{\mathbf{u}}_1,\ \mathbf{E}_2 = \mathbf{U}_2 - \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\mathbf{U}_1$$

which is the same result as that obtained by Shinozuka and Zhang [2] using a different approach. Further, if Eq. (13) is applied to $\mathbf{U}_1|_1$, then $\mathbf{C}_{12}{}' = \mathbf{C}_{11}$ and $\mathbf{U}_1|_1 = \bar{\mathbf{u}}_1$ as expected.

In addition, if $\bar{\mathbf{u}}_1$ is replaced by the random variate $\mathbf{U}_1$, then it may be verified that $\mathbf{U}_2^e$ and $\mathbf{E}_2$ are orthogonal in the probabilistic sense,[3] i.e.

$$E[(\mathbf{U}_2^e - \boldsymbol{\mu}_2^e)(\mathbf{E}_2 - \mathbf{e}_2)'] = 0 \tag{14}$$

where

$$\mathbf{U}_2^e = \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\mathbf{U}_1,\ \boldsymbol{\mu}_2^e = E[\mathbf{U}_2^e] = \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\boldsymbol{\mu}_1,$$

$$e_2 = E[\mathbf{E}_2] = \boldsymbol{\mu}_2 - \mathbf{C}_{12}'\mathbf{C}_{11}^{-1}\boldsymbol{\mu}_1$$

and if $\mathbf{U}_1$ and $\mathbf{U}_2$ are jointly Gaussian random variates, then $\mathbf{U}_2^e$ and $\mathbf{E}_2$ are statistically independent. This independence property is essential for simulation based on Eq. (13) as noted by Shinozuka and Zhang [2]. In view of the identity of Eqs. (8) and (9) for $\boldsymbol{\mu}_2|_1$ and $\mathbf{C}_{22}|_1$ with Eq. (7), simulation based on the conditional random field $\mathbf{U}_2|_1$ of Eq. (13) and simulation based on the conditional probability density function Eq. (6) are identical for Gaussian random variates. The reader is referred to the paper by Shinozuka and Zhang [2] for a full discussion of this point.

## 3. Random field with fixed ends

The conditional random field $U_c(z)$ on (0,1) is required to satisfy the end conditions

$$U_c(0) = \bar{u}_1,\ U_c(1) = \bar{u}_2 \tag{15}$$

where $\bar{u}_1$ and $\bar{u}_2$ are given deterministic values. For analysis of $U_c(z)$ using the general results of Section 2, we take $z_1 = 0$, $z_2 = 1$, $z_3 = z$, and $z_4 = z'$, where $z$ and $z'$ are arbitrary points on (0,1). Further, for mathematical simplicity, we consider the variance-homogeneous case where the unrestricted random field $U(z)$ has constant variance $\sigma^2$ and its covariance function may be written as

$$C(z,z') = \sigma^2\rho(z,z')$$

where $\rho(z,z')$ is the correlation coefficient function. Then, from Eq. (5)

$$\mathbf{C}_{11} = \sigma^2\begin{bmatrix} 1 & \rho(0,1) \\ \rho(0,1) & 1 \end{bmatrix},\ \mathbf{C}_{12} = \sigma^2\begin{bmatrix} \rho(0,z) & \rho(0,z') \\ \rho(z,1) & \rho(z',1) \end{bmatrix}$$

$$\mathbf{C}_{22} = \sigma^2\begin{bmatrix} 1 & \rho(z,z') \\ \rho(z,z') & 1 \end{bmatrix},\ \mathbf{U}_1 = \begin{bmatrix} U(0) \\ U(1) \end{bmatrix},$$

$$\mathbf{U}_2 = \begin{bmatrix} U(z) \\ U(z') \end{bmatrix} \quad \boldsymbol{\mu}_1 = \begin{bmatrix} \mu(0) \\ \mu(1) \end{bmatrix},\ \boldsymbol{\mu}_2 = \begin{bmatrix} \mu(z) \\ \mu(z') \end{bmatrix},$$

$$\text{and } \bar{\mathbf{u}}_1 = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}. \tag{17}$$

These equations will be used in the general results of the previous section to derive the conditional probability density function and an estimation–simulation formula for the conditional random field.

### 3.1. Conditional probability density function

On further requiring that $u(z)$ be a Gaussian random field, from Eqs. (7) and (17), we find that

$$\boldsymbol{\mu}_{2|1} = \begin{bmatrix} \mu_c(z) \\ \mu_c(z') \end{bmatrix},\ \mathbf{C}_{22|1} = \begin{bmatrix} \sigma_c^2(z) & C_c(z,z') \\ C_c(z,z') & \sigma_c^2(z') \end{bmatrix} \tag{18}$$

where the conditional mean, variance, and covariance functions are given by

$$\mu_c(z) = \mu(z) + \phi_1(z)[\bar{u}_1 - \mu(0)] + \phi_2(z)[\bar{u}_2 - \mu(1)]$$

$$\sigma_c^2(z) = \sigma^2\left[1 - \frac{\rho^2(0,z) + \rho^2(z,1) - 2\rho(0,z)\rho(z,1)\rho(0,1)}{1-\rho^2(0,1)}\right]$$

$$C_c(z,z') = \sigma^2\left\{\rho(z,z') - \frac{\rho(0,z)\rho(0,z') + \rho(z,1)\rho(z',1)}{1-\rho^2(0,1)}\right.$$

$$\left. + \frac{\rho(0,1)[\rho(0,z')\rho(z,1) + \rho(0,z)\rho(z',1)]}{1-\rho^2(0,1)}\right\} \tag{19}$$

where

$$\phi_1(z) = \frac{\rho(0,z) - \rho(z,1)\rho(0,1)}{1-\rho^2(0,1)},$$

$$\phi_2(z) = \frac{\rho(z,1) - \rho(0,z)\rho(0,1)}{1-\rho^2(0,1)} \tag{20}$$

It can be verified that Eq. (19) satisfies

$$\mu_c(0) = \bar{u}_1,\ \mu_c(1) = \bar{u}_2,\ C_c(z',z) = C_c(z,z')$$

$$C_c(z,z) = \sigma_c^2(z),\ C_c(z,0) = 0,\ C_c(z,1) = 0 \tag{21}$$

as expected. Further, if the unrestricted random field $U(z)$ is covariance homogeneous, then

$$\rho(z,z') = \rho(|z-z'|),\ \text{and } \phi_2(z) = \phi_1(1-z) \tag{22}$$

[3] This is known as the normality condition in linear estimation theory.

We define the effective correlation length $\ell_\epsilon$ of the unrestricted random field by the relation

$$|\rho(z,z')| < \epsilon, \text{ for } |z - z'| > \ell_\epsilon \tag{23}$$

where $\epsilon \ll 1$ is specified. When the effective correlation length is much smaller than the unit interval length ($\ell_\epsilon \ll 1$), then Eq. (19) reduces to

$$\mu_c(z) = \mu(z) + \rho(0,z)[\bar{u}_1 - \mu(0)] + \rho(z,1)[\bar{u}_2 - \mu(1)] + \mathcal{O}(\epsilon)$$

$$\sigma_c^2(z) = \sigma^2[1 - \rho^2(0,z) - \rho^2(z,1) + \mathcal{O}(\epsilon^2)] \tag{24}$$

$$C_c(z,z') = \sigma^2[\rho(z,z') - \rho(0,z')\rho(0,z) - \rho(z',1)\rho(z,1) + \mathcal{O}(\epsilon)]$$

In this case, the fixed end constraints are seen to introduce corrections to the unrestricted field only near the ends.

### 3.2. Conditional estimation

By Eq. (17), we have

$$\mathbf{C}_{12}'\mathbf{C}_{11}^{-1} = \begin{bmatrix} \phi_1(z) & \phi_2(z) \\ \phi_1(z') & \phi_2(z') \end{bmatrix} \tag{25}$$

where $\phi_1(z)$ and $\phi_2(z)$ are defined by Eq. (20). Then, from Eq. (13), the conditional random variate is

$$\mathbf{U}_{2|1} = \begin{bmatrix} U_c(z) \\ U_c(z') \end{bmatrix} \tag{26}$$

where the conditional random field $U_c(z)$ is given by

$$U_c(z) = U(z) + \phi_1(z)[\bar{u}_1 - U(0)] + \phi_2(z)[\bar{u}_2 - U(1)] \tag{27}$$

The estimation coefficients $\phi_1(z)$ and $\phi_2(z)$ also may be interpreted as weight functions that account for the effect of the fixed end constraints on the conditional random field. From Eq. (27) the conditional mean, variance, and covariance functions are given by

$$\mu_c(z) = \mu(z) + \phi_1(z)[\bar{u}_1 - \mu(0)] + \phi_2(z)[\bar{u}_2 - \mu(1)]$$

$$\sigma_c^2(z) = \sigma^2[1 + \phi_1(z)^2 + \phi_2(z)^2 + 2\rho(0,1)\phi_1(z)\phi_2(z) - 2\rho(0,z)\phi_1(z) - 2\rho(z,1)\phi_2(z)]$$

$$C_c(z,z') = \sigma^2[\rho(z,z') + \phi_1(z)\phi_1(z') + \phi_2(z)\phi_2(z') + \rho(0,1)\phi_1(z)\phi_2(z') + \rho(0,1)\phi_2(z)\phi_1(z') - \rho(0,z')\phi_1(z) - \rho(0,z)\phi_1(z') - \rho(z',1)\phi_2(z) - \rho(z,1)\phi_2(z')] \tag{28}$$

It may be verified directly that $\phi_1(z)$ and $\phi_2(z)$ from Eq. (20) minimize the variance $\sigma_c^2(z)$ for all $z$ and then Eq. (19) of the PDF approach follow from Eq. (28) as expected from the general results of the previous section. If the effective correlation length $\ell_\epsilon$ defined by Eq. (23) is much smaller than the unit interval length ($\ell_\epsilon \ll 1$), then Eq. (27) reduces to

$$U_c(z) \simeq U(z) + \rho(0,z)[\bar{u}_1 - U(0)] + \rho(z,1)[\bar{u}_2 - U(1)] + \mathcal{O}(\epsilon) \tag{29}$$

and the effect of fixed end constraints again is confined to regions near the ends.

From a mathematical viewpoint, the variance minimization property of $\phi_1(z)$ and $\phi_2(z)$ defined by Eq. (20) does not preclude other choices for $\phi_1(z)$ and $\phi_2(z)$ in Eq. (27) subject to

$$\phi_1(0) = 1,\ \phi_1(1) = 0,\ \phi_2(0) = 0,\ \phi_2(1) = 1 \tag{30}$$

In particular, Elishakoff [3] proposed linear weight functions of the form

$$\phi_1(z) = 1 - z,\ \phi_2(z) = z \tag{31}$$

and obtained expressions for the conditional variance and covariance functions assuming a strictly homogeneous unrestricted random field $U(z)$ with zero mean. From Eq. (28), these results are extended to a variance homogeneous unrestricted random field $U(z)$ with nonhomogeneous mean. For modeling small shape imperfections along a beam, Elishakoff's method corresponds to giving the beam a small rigid body rotation and translation to meet the end conditions after the unrestricted random imperfection is formed in a particular realization. An example will be given to illustrate the difference between the minimum-variance weight functions and Elishakoff's linear weight functions.

## 4. Conditional series expansion

The general series expansion for a random field given by Zhang and Ellingwood [4] is extended here to include conditioning by fixed end constraints. This also generalizes previous results by Elishakoff [3] for a sine series expansion based on his linear weight functions. According to Zhang and Ellingwood [4], the 1D-1V random field $U(z)$ on (0,1) with mean function $\mu(z)$ and covariance function $C(z,z')$ defined by Eq. (1) can be expressed as the series

$$U(z) = \mu(z) + \sum_{n=0}^{\infty} c_n \nu_n h_n(z) \tag{32}$$

where $c_n$ are constant expansion coefficients, $\nu_n$ are zero-mean random variates, and $h_n(z)$ are a complete set of deterministic orthonormal basis functions, i.e.

$$\int_0^1 h_m(z)h_n(z)\mathrm{d}z = \delta_{mn} \tag{33}$$

where $\delta_{mn}$ is the Kronecker delta (unit matrix). The coefficients $c_n$ and the covariance matrix $C_{mn}$ for $\nu_n$ remain to be determined from the covariance function $C(z,z')$. From Eqs. (1) and (32) we have

$$C(z,z') = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_m c_n C_{mn} h_m(z) h_n(z') \text{ where } C_{mn} = E[\nu_m \nu_n] \quad (34)$$

from which, with Eq. (33), it follows that

$$c_m c_n C_{mn} = \int_0^1\int_0^1 C(z,z') h_m(z) h_n(z') \mathrm{d}z\mathrm{d}z' \quad (35)$$

Without loss of generality, we may require that the variates $\nu_n$ have unit variance, i.e. $C_{mn} = 1$ if $m = n$, in which case Eq. (35) gives

$$c_n = \left[\int_0^1\int_0^1 C(z,z') h_n(z) h_n(z') \mathrm{d}z\mathrm{d}z'\right]^{\frac{1}{2}} \quad (36)$$

and $C_{mn}$ follows from Eq. (35) with $c_n$ known.

Now, application of the series expansion Eq. (32) to the conditional random field $U_c(z)$ with fixed end constraints Eq. (15) gives

$$U_c(z) = \mu_c(z) + \sum_{n=0}^{\infty} c_n \nu_n h_n(z) \quad (37)$$

where the mean function $\mu_c(z)$ is given by Eq. (19) part 1. Further, in view of Eq. (21), uniform convergence of the series Eq. (37) at the ends of the interval requires that[4]

$$h_n(0) = h_n(1) = 0 \quad (38)$$

For the general case of arbitrary weight functions $\phi_1(z)$ and $\phi_2(z)$, on substituting Eq. (28) part 3, for the conditional covariance function into Eq. (35), we have

$$\begin{aligned} c_m c_n C_{mn} = \sigma^2[&Y_{mn} + \Phi_{1m}\Phi_{1n} + \Phi_{2m}\Phi_{2n} \\ &+ \rho(0,1)(\Phi_{1m}\Phi_{2n} + \Phi_{2m}\Phi_{1n}) - \Phi_{1m}\Psi_{1n} \\ &- \Phi_{1n}\Psi_{1m} - \Phi_{2m}\Psi_{2n} - \Phi_{2n}\Psi_{2m}] \end{aligned} \quad (39)$$

where

$$Y_{mn} = \int_0^1\int_0^1 \rho(z,z') h_m(z) h_n(z') \mathrm{d}z\mathrm{d}z'$$

$$\Psi_{1n} = \int_0^1 \rho(0,z) h_n(z)\mathrm{d}z,\ \ \Psi_{2n} = \int_0^1 \rho(z,1) h_n(z)\mathrm{d}z$$

$$\Phi_{1n} = \int_0^1 \phi_1(z) h_n(z)\mathrm{d}z,\ \ \Phi_{2n} = \int_0^1 \phi_2(z) h_n(z)\mathrm{d}z \quad (40)$$

For the minimum-variance case, on substituting Eq. (20) for $\phi_1(z)$ and $\phi_2(z)$, we have

$$\Phi_{1n} = \frac{\Psi_{1n} - \rho(0,1)\Psi_{2n}}{1-\rho(0,1)^2},\ \ \Phi_{2n} = \frac{\Psi_{2n} - \rho(0,1)\Psi_{1n}}{1-\rho(0,1)^2} \quad (41)$$

and then

$$c_m c_n C_{mn} = \sigma^2\left\{Y_{mn} - \frac{\Psi_{1m}\Psi_{1n} + \Psi_{2m}\Psi_{2n} - \rho(0,1)[\Psi_{1m}\Psi_{2n} + \Psi_{1n}\Psi_{2m}]}{1-\rho(0,1)^2}\right\} \quad (42)$$

which also follows directly from Eqs. (35) and (19). For $m = n$, when the variates $\nu_n$ have unit variance ($C_{nn} = 1$), from Eq. (42), we have

$$c_n = \sigma\left\{Y_{nn} - \frac{\Psi_{1n}^2 + \Psi_{2n}^2 - 2\rho(0,1)\Psi_{1n}\Psi_{2n}}{1-\rho(0,1)^2}\right\}^{\frac{1}{2}} \quad (43)$$

Further, for the covariance-homogeneous case, from Eqs. (22) and (40), if the basis functions $h_n(z)$ are symmetric about $z = (1/2)$, then $\Psi_{2n} = \Psi_{1n}$ and if they are antisymmetric about $z = (1/2)$, then $\Psi_{2n} = -\Psi_{1n}$. In either of these cases Eqs. (42) and (43) can be simplified.

For Elishakoff's case [3] of linear weight functions (31), the $\Phi$ integrals in Eq. (39) become

$$\Phi_{1n} = \int_0^1 (1-z) h_n(z)\mathrm{d}z,\ \ \Phi_{2n} = \int_0^1 z h_n(z)\mathrm{d}z \quad (44)$$

and $C_{mn}$ is given by Eq. (39). Elishakoff's results for a sine series expansion follow as a special case.

## 5. Examples

### 5.1. White noise

In order to illustrate the theoretical results obtained for one-dimensional, conditional random fields with fixed ends, let us first consider a zero-mean, strictly homogeneous, unrestricted random field $U(z)$ on (0,1) consisting of a pure white noise with variance, covariance, and correlation functions given by

$$\sigma_w^2(z) = \sigma_0^2\delta(0)\ \ C_w(z,z') = \sigma_0^2\delta(z-z')$$

$$\rho_w(z,z') = \frac{\delta(z-z')}{\delta(0)} \quad (45)$$

respectively, where $\delta(z)$ is the Dirac delta (symbolic) function and $\sigma_0^2$ is a constant parameter.

For the minimum-variance conditional case, from Eq. (19), (20) and (45), the conditional variance and covariance

[4] See Courant and Hilbert [5] for a theorem on the completeness and uniform convergence of series expansions in terms of eigenfunctions for boundary value problems which have admissibility conditions of the form Eq. (38).

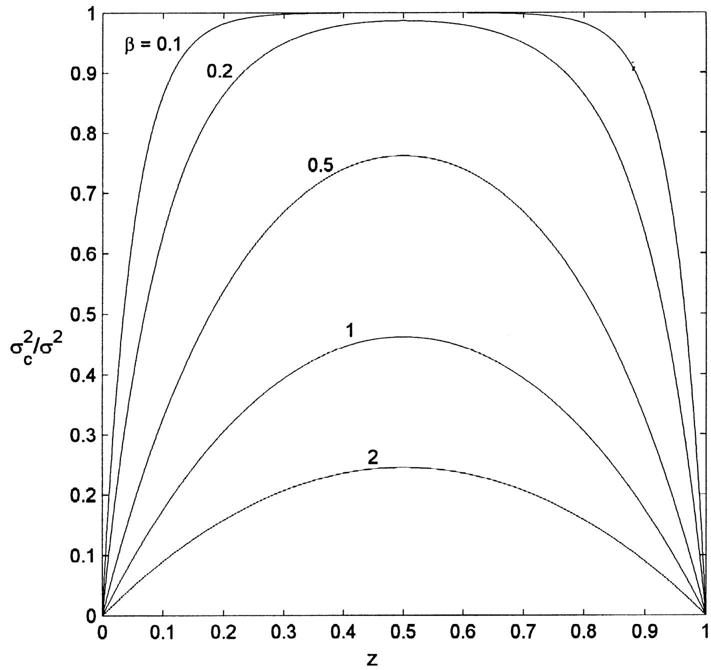


Fig. 1. Conditional variance function $\sigma_c^2(z)/\sigma^2$ for the exponential correlation function with various values of β in the minimum-variance case.

functions and $\phi_1(z)$, $\phi_2(z)$ can be written as

$$\sigma^2_{\mathrm{cmw}}(z) = \sigma_0^2\delta(0) \text{ for } 0 < z < 1$$

$$\sigma^2_{\mathrm{cmw}}(0) = \sigma^2_{\mathrm{cmw}}(1) = 0$$

$$C_{\mathrm{cmw}}(z,z') = \sigma_0^2\delta(z-z') \text{ for } 0 < z < 1,\ 0 < z' < 1$$

$$C_{\mathrm{cmw}}(0,z) = C_{\mathrm{cmw}}(z,1) = 0 \text{ for } 0 \le z \le 1$$

$$\phi_1(z) = \frac{\delta(z)}{\delta(0)},\ \phi_2(z) = \frac{\delta(1-z)}{\delta(0)} \tag{46}$$

Comparison of Eqs. (45) and (46) shows that conditioning has no effect on the variance and covariance functions except at the ends $z = 0$ and $z = 1$, as might be expected for pure white noise.

For the case of Elishakoff's linear weight functions [3] with $\phi_1(z)$ and $\phi_2(z)$ given by Eq. (31), using Eqs. (28) and (35), the conditional variance and covariance functions can be written as

$$\sigma^2_{\mathrm{clw}}(z) = 2\sigma_0^2\delta(0)(1 - z + z^2) \text{ for } 0 < z < 1$$

$$\sigma^2_{\mathrm{clw}}(0) = \sigma^2_{\mathrm{clw}}(1) = 0$$

$$C_{\mathrm{clw}}(z,z') = \sigma_0^2\delta(0)(1 - z - z' + 2zz') \text{ for } 0 < z < 1,$$

$$0 < z' < 1,\ z \ne z'$$

$$C_{\mathrm{clw}}(0,z) = C_{\mathrm{clw}}(z,1) = 0 \text{ for } 0 \le z \le 1 \tag{47}$$

Near $z = 0$ and $z = 1$, the variance $\sigma^2_{\mathrm{clw}}(z)$ is twice the unconditional variance and it diminishes to a minimum of 3/2 times the unconditional variance at $z = 1/2$. The value $2\sigma^2$ arises from the variance $\sigma^2$ of the unrestricted field at the point in question plus the variance $\sigma^2$ introduced by the (uncorrelated) end condition near the end as seen from Eq. (27). Thus, the conditional variance in the linear weight function case is considerably higher than that in the minimum-variance case. Further, for $0 < z < 1$, $0 < z' < 1$, and $z \ne z'$, the conditional covariance function is not zero in contrast to the unconditional case Eq. (45) and the minimum-variance conditional case Eq. (46).

### 5.2. Exponential correlation function

Next, we consider a zero-mean, covariance homogeneous, unrestricted random field $U(z)$ on (0,1) with constant variance $\sigma^2$ and exponential correlation coefficient function

$$\rho_e(z,z') = \exp\left[-\beta^{-1}|z-z'|\right] \tag{48}$$

where β is a correlation length parameter, since from Eq. (23), $\ell_\epsilon = -\beta\ln\epsilon$. Note that as $\beta \to 0, \rho(z,z')$ approaches the white noise limit Eq. (45).

For the minimum-variance case, using Eqs. (19), (20) and (48), the conditional variance and covariance functions and $\phi_1(z)$, $\phi_2(z)$ can be written as

$$\sigma^2_{\mathrm{cme}}(z) = 2\sigma^2(\sinh\beta^{-1})^{-1}\sinh\beta^{-1}(1-z)\sinh\beta^{-1}z$$

$$C_{\mathrm{cme}}(z,z') = 2\sigma^2(\sinh\beta^{-1})^{-1}\sinh\beta^{-1}(1-z)\sinh\beta^{-1}z',$$

for $(z \ge z')$

$$\phi_2(z) = (\sinh\beta^{-1})^{-1}\sinh\beta^{-1}z\ \ \phi_1(z) = \phi_2(1-z) \tag{49}$$

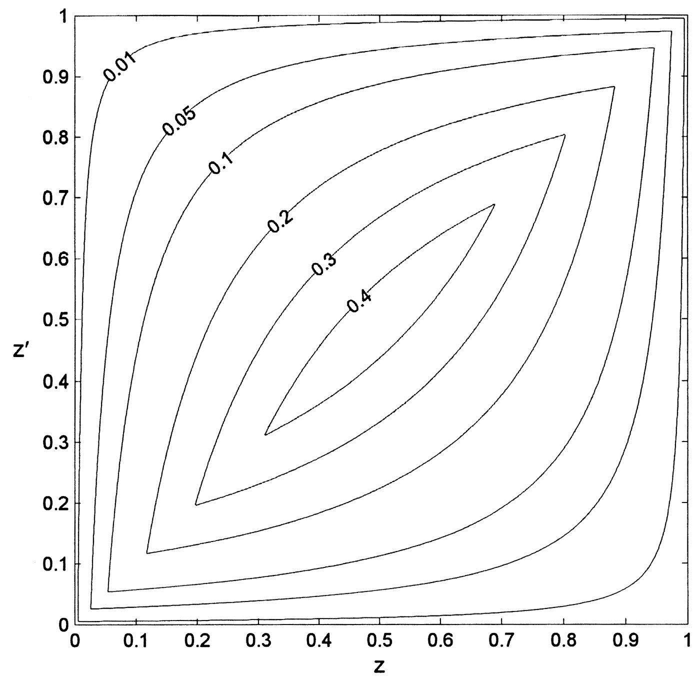


Fig. 2. Contours of the conditional covariance function $C_c(z,z')/\sigma^2$ for the exponential correlation function with $\beta = 1$ in the minimum-variance case.

The conditional variance function $\sigma^2_{\rm cme}(z)$ from Eq. (49) is shown in Fig. 1 for various values of β. As $\beta \to 0$, $\sigma^2_{\rm cme}(z)/\sigma^2$ smoothly approaches the white noise limit $\sigma^2_{\rm cmw}/\sigma^2$ given by Eq. (46) where $\sigma^2 = \sigma_0^2\delta(0)$ in both cases. Contours of the conditional covariance function $C_{\rm cme}(z,z')$ from Eq. (49) are shown in Figs 2 and 3 for $\beta = 1$ and $\beta = 0.1$, respectively. Fig. 4 shows the weight function $\phi_1(z)$ from Eq. (49). From all of these figures it can be seen that the effect of the fixed end constraints is confined to the neighborhood of the ends when β is small ($\beta < 0.1$) as expected from Eqs. (23), (24) and (29). For larger values of β the conditional variance and covariance functions are reduced from the unrestricted case for all values of $z$.

As an illustration of the conditional series expansion method of the preceding section, we consider a sine series expansion for the exponential correlation coefficient function

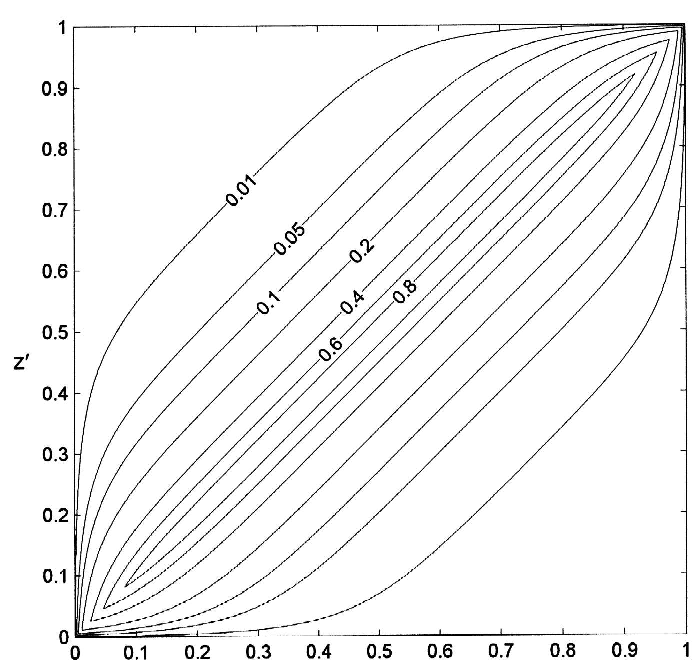


Fig. 3. Contours of the conditional covariance function $C_c(z,z')/\sigma^2$ for the exponential correlation function with $\beta = 0.1$ in the minimum-variance case.

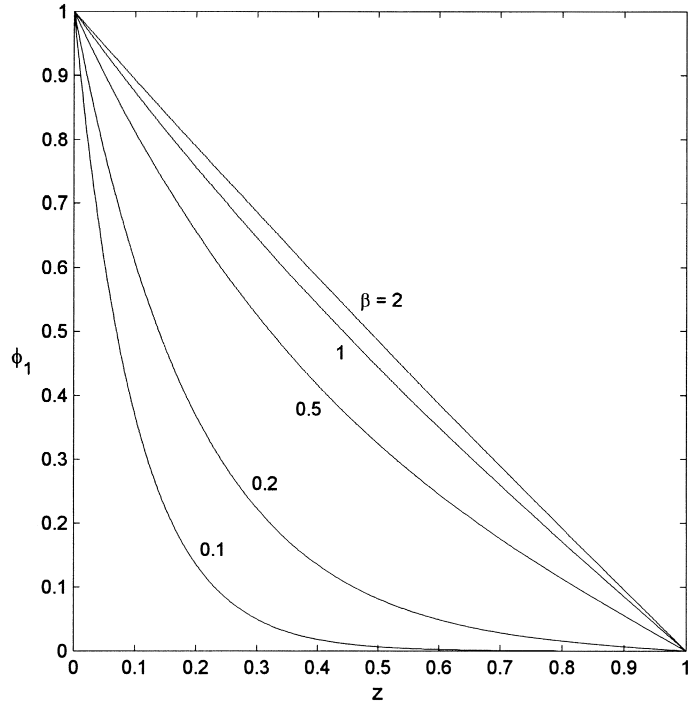


Fig. 4. End conditioning weight function $\phi_1(z)$ for the exponential correlation function with various values of β in the minimum-variance case.

in the minimum-variance case. With basis functions

$$h_n(z) = \sqrt{2}\sin n\pi z$$

by either Eqs. (40), (42), (43) and (49), or Eqs. (35), (36) and (49), after a lengthy derivation, we find that

$$C_{mn} = \delta_{mn},\ c_n = \sigma\left[\frac{2\beta}{1 + n^2\pi^2\beta^2}\right]^{\frac{1}{2}} \tag{50}$$

The fact that the same result is obtained by both approaches provides a check on the formulas developed for conditional series expansion. Then, from Eqs. (34) and (50), we have the following series expansions for the covariance and variance functions:

$$C_{\text{cme}}(z, z') = 4\beta\sigma^2 \sum_{n=1}^{\infty} \frac{\sin n\pi z \sin n\pi z'}{1 + n^2\pi^2\beta^2},$$
$$\sigma^2_{\text{cme}}(z) = 2\beta\sigma^2 \sum_{n=1}^{\infty} \frac{1 - \cos 2n\pi z}{1 + n^2\pi^2\beta^2} \tag{51}$$

The series for $\sigma^2_{\text{cme}}(z)$ also can be obtained directly by expansion of Eq. (49), in a Fourier cosine series, thereby providing a further check on the derivation of Eq. (50). The error in $\sigma^2_{\text{cme}}(z)$ from truncating the original series (37), and hence the series (51), at $N$ terms is given in Table 1. The convergence is more rapid as β increases as expected from the coefficients of the series (51). Further, the convergence is slower near $z = 0$ and $z = 1$ since the sine series for the first derivative of $\sigma^2_{\text{cme}}(z)$, obtained by differentiation of Eq. (51), gives a value of zero for the first derivative at $z = 0$ and $z = 1$, whereas the actual conditional variance function Eq. (49) has a nonzero first derivative at these points. Considering the extended interval $-1 \le z \le 1$, the cosine series Eq. (51) for $\sigma^2_{\text{cme}}(z)$ represents an even function, whereas the actual conditional variance function Eq. (49) on the extended interval is an odd function of $z$. The extended even function represented by the cosine series whas a discontinuity in its first derivative at $z = 0$ and $z = 1$. Thus, the sine series for the first derivative of $\sigma^2_{\text{cme}}(z)$ obtained from Eq. (51) does not converge uniformly in an interval containing $z = 0$ or $z = 1$. It should be kept in mind that the series (51) for the covariance and variance

Table 1
Error in series for $\sigma^2_{\text{cme}}(z)$ truncated at $N$ terms

| | $N = 15$ (%) | $N = 51$ (%) | $N = 101$ (%) | $N = 501$ (%) |
|---|---|---|---|---|
| $z = 0.5$, β = 0.1: | − 12 | − 4 | − 2 | − 0.4 |
| $z = 0.5$, β = 2.0: | − 3 | − 0.8 | − 0.4 | − 0.1 |
| $z = 0.05$, β = 0.1: | − 17 | − 6 | − 3 | − 0.6 |
| $z = 0.05$, β = 2.0: | − 11 | − 4 | − 2 | − 0.4 |

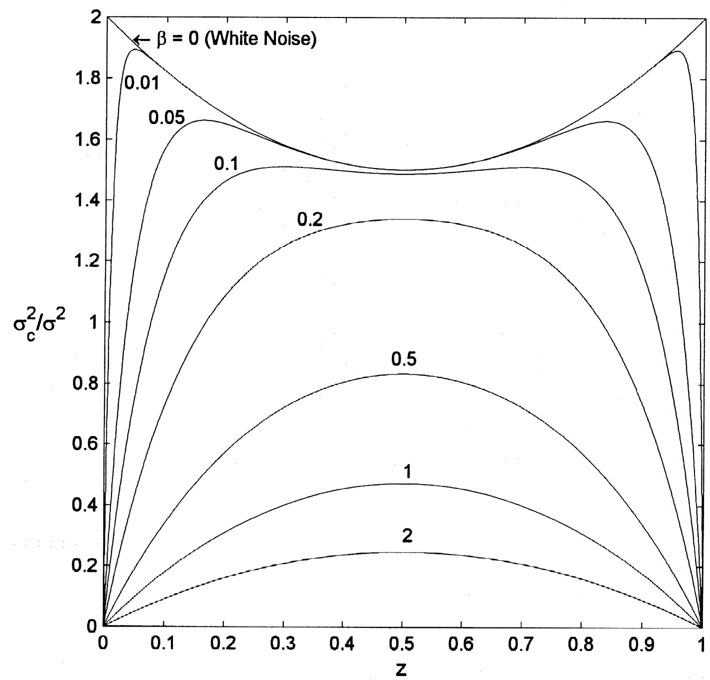


Fig. 5. Conditional variance function $\sigma_c^2(z)/\sigma^2$ for the exponential correlation function with various values of β in the linear weight function case.

functions follow from the original sine series for the random function $U(z)$ obtained from Eq. (32) which is uniformly convergent as previously noted.

For the case of Elishakoff's linear weight functions [3] with $\phi_1(z)$ and $\phi_2(z)$ given by Eq. (31), the conditional variance function $\sigma_{cle}^2(z)$ calculated from Eq. (28) is shown in Fig. 5 for various values of β. For $\beta > 1$, $\sigma_{cle}^2(z)$ is nearly identical to $\sigma_{cme}^2(z)$ of the minimum-variance case shown in Fig. 1 as expected since $\phi_1(z) \simeq 1 - z$ for $\beta > 1$ as seen from Eq. (49) and Fig. 4. However, for $\beta < 0.5$ there is a considerable difference between the two cases. As $\beta \rightarrow 0$, $\sigma_{cle}^2(z)/\sigma^2$ smoothly approaches the white noise limit

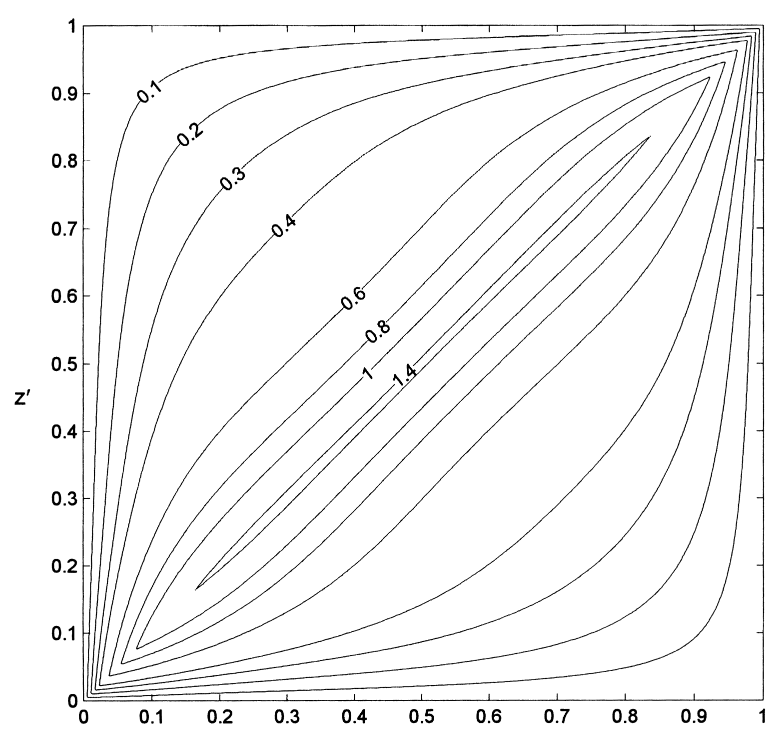


Fig. 6. Contours of the conditional covariance function $C_c(z,z')/\sigma^2$ for the exponential correlation function with $\beta = 0.1$ in the linear weight function case.

$\sigma^2_{\rm clw}(z)/\sigma^2$ given by Eq. (47) where $\sigma^2 = \sigma_0^2\delta(0)$ in both cases. For $\beta = 0.1$ contours of the $C_{\rm cle}(z,z')$ calculated from Eq. (19) part 3, are shown in Fig. 6. These results differ significantly from those of the minimum-variance case shown in Fig. 3. Again, for $\beta > 1$ the conditional covariance function $C_{\rm cle}(z,z')$ of the linear weight function case is nearly identical to $C_{\rm cme}(z,z')$ of the minimum-variance case.

## 6. Summary and conclusions

We have derived expressions for the mean, variance, and covariance functions for a one-dimensional (1D), uni-variate (1V) random field conditioned by deterministic fixed end values in terms of the known mean, variance, and correlation coefficient functions of an unrestricted, variance-homogeneous, Gaussian random field using a known relation for the conditional probability density function [1]. Also, we have derived a functional relation between the unrestricted random field and the corresponding conditional random field based on optimum (minimum-variance) linear estimation theory [1]. The mean, variance and covariance functions derived from the functional representation of the conditional random field are the same as those derived from the conditional probability density function as pointed out by Shinozuka and Zhang [2]. Further, we have generalized results obtained by Elishakoff [3] based on enforcing end conditions by linear weight functions. In addition, we have obtained formulas for the coefficients in a series expansion for the conditional random field using the results of Zhang and Ellingwood [4].

Examples have been given to illustrate the effect of conditioning by specification of zero values for the random field at the ends of the interval. Comparison is made between results from the present minimum-variance method and the linear weight function method of Elishakoff [3]. Results from the two methods agree for random fields with large correlation lengths on the order of the interval length. However, the results differ significantly for random fields with correlation lengths much shorter than the interval length. In applications, the choice of a method to enforce end constraints will depend on the physical characteristics of the random field and the idealization of the actual problem. Also, the conditional series expansion method has been illustrated by an example for which convergence of the series for the conditional variance is investigated.

The results presented here have direct application to the characterization and simulation of conditional 1D-1V random fields to model imperfections with fixed ends in the probabilistic stability analysis of columns and arches. Other applications appear to be possible. A slight generalization of our results would allow inclusion of the effect of measurements of the random field at discrete points. Generalization of the present results to random fields in two and three dimensions also would be of interest.

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