## Random Vibration Analysis of Dynamic Vehicle-Bridge Interaction Due to Road Unevenness

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**Abstract:** Vehicle-bridge interaction has been studied for a long time to investigate the structural behavior of bridges and vehicle ride comfort. An original frequency domain method is presented where the vehicle-bridge interaction problem is solved in a frame of reference that moves with the vehicle. The Fourier transform of the interaction force is computed directly from the vehicle compliance and bridge compliance, without requiring any iterations. The method is particularly useful when a closed-form solution of the bridge compliance is available, as in the case of a simply supported Euler-Bernoulli beam model for the bridge. The solution is, therefore, well-suited for parametric studies on the bridge and vehicle response characteristics and offers a reference for more detailed models of the bridge and the vehicle or more complicated bridge configurations (e.g., continuous beam on multiple supports). The frequency domain approach also leads to enhanced physical understanding, because it shows how the interaction force decomposes into a term resulting from the dynamic response of the bridge to the constant moving load component and a term because of road surface unevenness. An efficient solution procedure based on random vibration analysis is presented, which allows for the computing of the statistical characteristics of the bridge and vehicle response from the power spectral density function of the unevenness. The procedure is validated by means of Monte Carlo simulation results for the case where the passage of a heavy vehicle on a highway bridge is considered. **DOI: 10.1061/(ASCE)EM.1943-7889.0000386.** © *2012 American Society of Civil Engineers*.

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## Introduction

Vehicle-bridge interaction has been a subject of significant research for a long time. The aim of these studies is to investigate the structural behavior of bridges under moving vehicles, as well as the ride comfort of vehicles traversing a bridge. Dynamic vehicle-bridge interaction results in an increase or decrease of the bridge deformations, which is described by the dynamic amplification factor (DAF) that reflects how many times the constant load must be multiplied to cover additional dynamic effects (Frýba 1996). Although the additional dynamic loads usually do not lead to major bridge failures, they contribute to a continuous degradation of the bridge, increasing the necessity of regular maintenance (Cebon 1999).

A historical overview of research on bridge dynamics in Europe, the United States, and Asia is given by Frýba (1996). Early research has led to closed-form or analytical expressions for simplified cases, e.g., considering a simply supported Euler-Bernoulli beam model for the bridge and a moving load model for the vehicle (Biggs 1964; Frýba 1999). In the case where the vehicle is represented by a moving mass or moving oscillator, the vehicle-bridge interaction problem requires the simultaneous solution of a set of coupled partial differential equations governing the motion of the bridge and the vehicle. For a moving mass traversing a simply supported Euler-Bernoulli beam, Biggs (1964) has reformulated the set of coupled differential equations into a single ordinary differential equation with time-dependent coefficients that only considers the fundamental mode of the bridge. An overview of solutions to the moving load, moving mass, and moving oscillator problem is given by Yang et al. (2000). Furthermore, Yang et al. (2000) present a semianalytical solution of the moving oscillator problem, where the response of the coupled system is formulated in terms of an integral equation that allows for a straightforward numerical solution. Numerical results are presented for the case of a string and a simply supported beam.

A large number of studies have also proposed numerical solution procedures as a solution of the vehicle-bridge interaction problem. A first approach consists in an iterative solution of the equations of motion of the vehicle and bridge (Green et al. 1995; Henchi et al.1998; Liu et al. 2009). A drawback is the computational effort involved with the iterative scheme. This can be avoided by eliminating the vehicle-bridge interaction forces from both sets of equations. When the finite-element method is used to solve the problem, this results in a single stiffness matrix, damping matrix, and mass matrix for the coupled system (Kim et al. 2005), which may no longer be symmetric, however. Alternatively, Yang and Lin (1995) and Yang and Yau (1997) have developed a vehiclebridge interaction element that allows computing the response with a relatively small number of iterations. An experimental validation of a three-dimensional model for vehicle-bridge interaction based on simultaneous measurements of the vehicle response and strains measured on a 40-m span, steel plate girder bridge has been presented by Kim et al. (2005). Similar experiments have been performed by Brady et al. (2006) to investigate the DAF for a single

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vehicle traversing a bridge, as well as for two vehicles crossing the bridge simultaneously.

A distinction can be made between two excitation mechanisms for dynamic vehicle-bridge interaction. First, the vehicle is excited by the dynamic bridge deflection at the contact point between the vehicle and the bridge. Even when the road surface is perfectly smooth, this will lead to additional load effects and a modification of the bridge response compared with the moving load case. Second, irregularities in the bridge roadway surface will contribute to the vehicle loads as well and will therefore also affect the response of the coupled system. Track and road unevenness are often modeled as a stationary Gaussian random field characterized by its power spectral density (PSD) function (ORE 1971; ISO 1991; Schiehlen 2009). Because of the motion of the vehicle on the bridge, the vehicle load and bridge response are nonstationary random processes, of which the second-order statistical characteristics are described by the nonstationary autocorrelation function (ACF). The problem of vehicle-bridge interaction induced by random track unevenness has mainly been solved using Monte Carlo simulations of the deterministic problem for a given ensemble of realizations of the track unevenness (Green et al. 1995; Xia et al. 2001). To obtain accurate second-order statistics of the dynamic bridge response, however, a large number of samples may be required (Lu et al. 2009; Zhang et al. 2010), because road surface unevenness profiles derived from the same PSD may yield a significantly different value for the DAF (Brady et al. 2006). Recently, Lu et al. (2009) and Zhang et al. (2010) have shown how the pseudoexcitation method obtains the nonstationary second-order statistical characteristics of the bridge dynamic response, based on the deterministic response to a set of so-called pseudoexcitations. The precise integration method is used to obtain an efficient iterative solution (Zhang et al. 2010).

An original frequency domain method is presented where the vehicle-bridge interaction problem is solved in a frame of reference that moves with the vehicle (Clouteau et al. 2001). The method requires calculation of vehicle and bridge compliance, which represents, for each system, the ratio between the load applied at the contact point and the corresponding displacement. The method is particularly appealing where closed-form solutions of the compliance functions are available, as in the case of a quarter-car model for the vehicle and a simply supported Euler-Bernoulli beam model for the bridge (Lombaert and Conte 2011). The solution is wellsuited for parametric studies on the bridge and vehicle response characteristics and provides a reference solution for more detailed models of the vehicle and the bridge or more complicated bridge configurations (e.g., continuous beam on multiple supports). The interaction forces are computed directly without requiring any iterations. Moreover, the method provides an enhanced insight in the contribution of road unevenness to the interaction force and the dynamic bridge response. In this way, the current study complements previous research aimed at the development of numerical methods to couple more advanced vehicle and bridge models. Finally, the proposed solution procedure for the interaction force also allows for a very efficient computation of the statistical characteristics of the bridge dynamic response based on the PSD function of the unevenness. The procedure is based on earlier developments for the calculation of the dynamic axle loads of a train traversing a track with random track unevenness (Lombaert and Degrande 2009).

The outline of this paper is as follows. In the subsequent section, the bridge response to a moving concentrated load with a given time-varying intensity f(t) is computed by modal superposition. In the derivation, a simply supported beam model is considered for the bridge. This model is valid for a large number of single span

road and railway bridges and has the advantage of yielding closedform solutions. These closed-form solutions have not been included for reasons of conciseness but can be found in Lombaert and Conte 2011. The solution by modal superposition is general, however, and can be applied to any bridge model or bridge configuration. Next, the time-varying intensity f(t) of the vehicle load is obtained by solving the dynamic vehicle-bridge interaction problem. Finally, the case of random road unevenness is considered and expressions are derived for the nonstationary second-order statistical characteristics of the interaction force f(t) and the bridge response. The solution is illustrated with a numerical example that considers the passage of a heavy vehicle on a single-span highway bridge with random road unevenness.

## Bridge Response to a Concentrated Moving Load

The response of the bridge to a moving concentrated load with a given time-varying intensity f(t) is computed. A simply supported Euler-Bernoulli beam model is used for the bridge, and therefore the vertical displacement field y(x, t) is governed by the following partial differential equation of motion (Clough and Penzien 1975):

$$EI\frac{\partial^4 y}{\partial x^4} + c_s I\frac{\partial^5 y}{\partial x^4 \partial t} + \bar{m}\frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} = p(x,t)$$
(1)

where E = Young's modulus, I = moment of inertia of the beam cross section, EI = bending stiffness,  $\bar{m}$  = mass per unit length, and p(x,t) = distributed vertical loading on the beam. Both EI and  $\bar{m}$  are assumed constant herein. The second and fourth term on the left-hand-side of Eq. (1) represent viscous damping forces, where  $c_s$  is the viscous resistance to the strain rate and c is the viscous resistance to the vertical velocity.

In the case of a moving concentrated load with a given timevarying intensity f(t) and speed v, the load p(x, t) can be expressed as

$$p(x,t) = \begin{cases} \delta(x - \upsilon t)f(t), & 0 \le t \le t_d \\ 0, & t > t_d \end{cases}$$
(2)

where  $t_d = L/v$  is the time required for the load to cross the bridge of length *L*. Eq. (1) is now solved for the load p(x, t) in Eq. (2) using modal superposition. A transformation is made from the displacement coordinates y(x, t) to the modal coordinates  $z_n(t)$ 

$$y(x,t) = \sum_{n=1}^{\infty} \phi_n(x) z_n(t)$$
(3)

where  $\phi_n(x)$  = undamped deflection mode shape *n* in the case of undamped free vibration. For a simply supported beam, the undamped natural frequencies  $\omega_n$ , vibration mode shapes  $\phi_n(x)$ , and modal damping ratios  $\xi_n$  are given by (Clough and Penzien 1975)

$$\omega_n = \frac{n^2 \pi^2}{L^2} \sqrt{\frac{EI}{\bar{m}}}, \qquad \phi(x) = \sin\left(\frac{n\pi x}{L}\right),$$

$$\xi_n = c/(2\bar{m}\omega_n) + c_s \omega_n/E, \qquad n = 1, 2, \dots$$
(4)

Taking advantage of the orthogonality properties of the mode shapes, the following system of uncoupled ordinary differential equations is obtained for the modal coordinates by introducing the transformation of Eq. (3) in Eq. (1)

$$\ddot{z}_n(t) + 2\xi_n \omega_n \dot{z}_n(t) + \omega_n^2 z_n(t) = f_n(t)$$
(5)

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where a superimposed dot = differentiation with respect to time and  $f_n(t)$  = modal load for the moving load in Eq. (2)

$$f_n(t) = \frac{1}{M_n} \int_0^L \phi_n(x) p(x, t) dx = \begin{cases} \frac{1}{M_n} \phi(vt) f(t), & 0 \le t \le t_d n \\ 0, & t > t_d \end{cases}$$
(6)

where  $M_n = \text{modal mass } \overline{m}L/2$ . The modal response  $z_n(t)$  is found from the modal impulse response function  $h_n(t)$  by means of the Duhamel's integral

$$z_n(t) = \int_0^{\min(t,t_d)} h_n(t-\tau) \frac{1}{M_n} \phi_n(\upsilon\tau) f(\tau) d\tau$$
(7)

The substitution of Eq. (7) in Eq. (3) provides the vertical displacement response y(x, t) at any position *x*.

An alternative expression is now derived where the modal response  $z_n(t)$  is computed from the Fourier transform  $F(\omega)$  of the moving load. The following convention is chosen for the forward and inverse Fourier transform, respectively:

$$\begin{cases} F(\omega) = \int_{-\infty}^{+\infty} f(t) \exp(-i\omega t) dt \\ f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \exp(i\omega t) d\omega \end{cases}$$
(8)

In the following, uppercase letters will be used for Fourier transforms of functions denoted by lowercase letters.

Eq. (8) is used to rewrite  $f(\tau)$  in Eq. (7) in terms of its Fourier transform  $F(\omega)$ . Switching the orders of integration leads to

$$z_n(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[ \int_0^{\min(t,t_d)} h_n(t-\tau) \frac{1}{M_n} \phi_n(\upsilon\tau) \exp(i\omega\tau) d\tau \right] \\ \times F(\omega) d\omega \tag{9}$$

Comparison with Eq. (7) shows that the term between the square brackets is the modal response  $z_n(t)$  to a moving concentrated load with harmonic time-varying intensity  $f(\tau) = \exp(i\omega\tau)$ . This bracketed term is denoted as  $g_n(t, -\omega)$ , where the minus sign is because of the convention in Eq. (8) assumed for the forward Fourier transform. Eq. (7) for the modal response  $z_n(t)$  is now rewritten as

$$z_n(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g_n(t, -\omega) F(\omega) d\omega \qquad (10)$$

In the case of a simply supported beam, a closed-form solution for  $g_n(t, -\omega)$  is obtained from Eq. (7) by means of the modal impulse response function  $h_n(t)$ , the modal mass  $M_n = \bar{m}L/2$ , and the undamped mode shape  $\phi_n(x)$  in Eq. (4).

## **Dynamic Vehicle-Bridge Interaction**

## General Formulation

In this section, the vehicle-bridge interaction is considered and the dynamic vehicle load is computed. The time history of the total vehicle load  $f_{st} + f(t)$  applied to the bridge is decomposed into the constant moving load  $f_{st}$ , corresponding to the weight of the vehicle, and the unknown variable moving load f(t) resulting from dynamic vehicle-bridge interaction (Fig. 1). A compliance formulation in a frame of reference that moves with the vehicle load f(t) (Clouteau et al. 2001). Assuming perfect contact between the vehicle and the bridge allows the following compatibility equation at any time  $0 \le t \le t_d$  when the vehicle is on the bridge:

$$u_v(t) = u_b(t) + r'(t), \qquad 0 \le t \le t_d$$
(11)



**Fig. 1.** Deflected bridge shape y(x, t) (thin black line), bridge shape with unevenness y(x, t) + r(x) (thick black line), assumed convention for the constant moving load  $f_{st}$  and the variable moving load f(t)

where  $u_v(t)$  and  $u_b(t) \equiv y(vt, t)$  are the displacement of the vehicle and the bridge, respectively, at the moving contact point x = vtbetween the vehicle and the bridge, while r'(t) is the unevenness r(x) at the contact point (Fig. 1). The compatibility Eq. (11) is written for a single contact point and is, therefore, valid for any single-axle vehicle model. In the case where a vehicle model with multiple axles is considered, Eq. (11) needs to be written for every contact point between the vehicle and the bridge.

Prior to its arrival on the bridge  $(t \le 0)$  and after its passage on the bridge  $(t \ge t_d)$ , the vehicle is assumed to travel on an uneven, rigid road pavement. In this way, the vehicle response builds up to a stationary level before the vehicle enters the bridge and effects of dynamic vehicle-bridge interaction can be clearly recognized. The corresponding compatibility equation is obtained by omitting the term  $u_b(t)$  in Eq. (11) for  $t \le 0$  and  $t \ge t_d$  or, equivalently, by assigning a zero value to  $u_b(t)$  for the same values of t.

The compatibility Eq. (11) is transformed into an integral equation for the variable moving load f(t) by rewriting  $u_v(t)$  and  $u_b(t)$  in terms of f(t). The functions vehicle response  $u_v(t)$  and bridge response  $u_b(t)$  to the variable moving load f(t) are the vehicle and bridge compliance, respectively.

When a linear vehicle model is used, the impulse response function  $c_v(t)$  of the vehicle can be used to compute  $u_v(t)$  from f(t) by means of the following Duhamel integral:

$$u_{\nu}(t) = -\int_{-\infty}^{t} c_{\nu}(t-\tau)f(\tau)d\tau \qquad (12)$$

where  $c_v(t - \tau)$  = vehicle response at the contact point at the time *t* because of a unit impulse load at time  $\tau$ . The vehicle compliance  $c_v(t - \tau)$  only depends on the difference  $t - \tau$ , as a linear time-invariant system is considered. The minus sign in Eq. (12) is because of the assumed convention that f(t) is positive when acting on the bridge in the upward direction (Fig. 1).

The bridge response  $u_b(t)$  at the contact point x = vt between the vehicle and the bridge is decomposed into the response to a moving load with a constant intensity  $f_{st}$  and a moving load with time-varying intensity f(t). The response  $u_{st}(t)$  to the constant moving load  $f_{st}$  is computed considering a load  $F(\omega) = f_{st}2\pi\delta(\omega)$  in Eq. (10) for the modal coordinates and a subsequent evaluation of Eq. (3) at the position x = vt of the moving contact point. The response to the variable moving load f(t) is calculated from the bridge compliance that relates, by definition, the variable moving load f(t) to the response at the moving point where the load is applied. Because of the motion of the load, the relationship between the load f(t) and the displacement at the (moving) contact point is time variant. The bridge compliance is, therefore, denoted by  $c_b(t, \tau)$  as a general time-variant system. This leads to the following expression for the bridge displacement  $u_b(t)$ :

$$u_{\upsilon}(t) = \int_{-\infty}^{t} c_b(t,\tau) f(\tau) d\tau + u_{st}(t)$$
(13)

Introducing Eqs. (12) and (13) for the vehicle compliance and the bridge compliance, respectively, in the compatibility Eq. (11) leads to the following integral equation for the interaction force f(t):

$$\int_{-\infty}^{t} [c_{v}(t-\tau) + c_{b}(t,\tau)]f(\tau)d\tau = -u_{st}(t) - r'(t)$$
(14)

The term between the square brackets represents the sum of the vehicle and bridge compliance and will be denoted as  $c_{vb}(t, \tau)$  in the following equation when the statistical characteristics of the interaction force and the bridge response are computed:

$$c_{vb}(t,\tau) = c_v(t-\tau) + c_b(t,\tau)$$
(15)

Eq. (14) shows that the interaction force f(t) is generated by the combined excitation because of the displacement  $u_{st}(t)$  at the contact point that results from the moving constant load  $f_{st}$ and the unevenness r'(t).

Eq. (14) is formulated in the frequency domain as follows. First, the upper limit t of the integral with respect to  $\tau$  is extended to  $+\infty$ , exploiting the causality of  $c_v(t-\tau)$  and  $c_b(t,\tau)$ . Second, f(t) is rewritten as the inverse Fourier transform of  $F(\omega)$ . A switch of the order of integration allows for the derivation of the following expression:

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} [C_v(\omega) \exp(i\omega t) + C_b(t, -\omega)] F(\omega) d\omega = -u_{st}(t) - r'(t)$$
(16)

Third, a Fourier transform with respect to t is performed, finally leading to

$$C_{\nu}(\omega')F(\omega') + \frac{1}{2\pi} \int_{-\infty}^{+\infty} C_b(\omega', -\omega)F(\omega)d\omega = -U_{st}(\omega') - R'(\omega')$$
(17)

Once a numerical solution for  $F(\omega)$  is obtained, the bridge response is computed from the modal response to a moving concentrated load with harmonic time-varying intensity in Eq. (10) and the modal superposition in Eq. (3).

## Vehicle Compliance

The vehicle compliance  $C_v(\omega)$  is the frequency domain ratio between the displacement at the contact point  $U_v(\omega)$  and the interaction force  $F(\omega)$ . For a single-axle vehicle model consisting of a lumped mass  $m_v$  supported by a spring and a dashpot with characteristics  $k_v$  and  $c_v$ , respectively, the following expression is derived for vehicle compliance  $C_v(\omega)$ :

$$C_{v}(\omega) \equiv -\frac{U_{v}(\omega)}{F(\omega)} = -\frac{k_{v} + i\omega c_{v} - m_{v}\omega^{2}}{(k_{v} + i\omega c_{v})m_{v}\omega^{2}}$$
(18)

At limiting low frequencies  $\omega \to 0$ , the vehicle compliance  $C_v(\omega)$  reduces to that of a simple rigid mass  $m_v$  (without spring and dashpot), namely  $C_v = -1/m_v\omega^2$ . This stems from the fact that a low-frequency base excitation or, equivalently, a long wavelength excitation of the vehicle does not induce any deformation in the spring and the dashpot. At zero frequency  $\omega = 0$ , the vehicle compliance  $C_v(\omega)$  tends to  $-\infty$ , because, when the vehicle is considered free from the bridge, the displacement of the contact point, because of a constant load  $f(t) = f_{st}$ , is unbounded. As a result, Eq. (17) can only be formulated for  $\omega' > 0$ .

## Bridge Compliance

The bridge compliance  $c_b(t, \tau)$  represents the bridge response at the moving contact point x = vt and at time t because of a unit impulse force applied at time  $\tau$  at the corresponding position  $x = v\tau$  of the contact point. The bridge compliance is computed from Eqs. (3) and (7)

$$c_b(t,\tau) = H(t_d - t) \sum_{n=1}^{N} \phi_n(\upsilon t) h_n(t - \tau) \frac{1}{M_n} \phi_n(\upsilon \tau) H(\tau) \quad (19)$$

where the Heaviside functions  $H(t_d - t)$  and  $H(\tau)$  ensure that  $u_b(t) = 0$  for  $t \le 0$  and  $t \ge t_d$ , as the vehicle is assumed to travel on a rigid road pavement prior to arriving and after leaving the bridge.

This expression is now used to obtain a formulation of the bridge compliance in the frequency domain. A double-forward Fourier transform with respect to t and  $\tau$  is applied to Eq. (19). Adjusting the integration limits in accordance with the Heaviside functions in Eq. (19) allows the double forward Fourier transform as follows:

$$C_b(\omega',\omega) = \sum_{n=1}^N \int_0^{t_d} \phi_n(\upsilon t) \left[ \frac{1}{M_n} \int_0^t h_n(t-\tau) \phi_n(\upsilon \tau) \exp(-i\omega\tau) d\tau \right] \\ \times \exp(-i\omega' t) dt$$
(20)

where the bracketed term = previously defined modal response  $g_n(t, \omega)$  to a moving concentrated load with harmonic time-varying intensity  $\exp(-i\omega t)$ .

## **Random Road Unevenness**

#### Autocorrelation Function of the Road Unevenness

In the following, the vehicle-bridge interaction problem is solved for the case where road unevenness with a stochastic character is present. It is assumed that the unevenness r(x) is a uniformly modulated random field

$$r(x) = c(x)\bar{r}(x) \tag{21}$$

where c(x) = modulation function and  $\bar{r}(x) =$  underlying homogeneous random field. The PSD  $\Phi_{\bar{r}\bar{r}}(k_x)$  of the underlying homogeneous random field  $\bar{r}(x)$  is usually expressed in terms of the wave number  $k_x = 2\pi/\lambda_x$ , where  $\lambda_x$  is the wavelength of the unevenness. The corresponding time variation of the excitation r'(t) at the moving contact point between the vehicle and the bridge is determined by the vehicle speed v as

$$c'(t)\overline{r}'(t) = c(\upsilon t)\overline{r}(\upsilon t) \tag{22}$$

where c'(t) = c(vt) and  $\bar{r}'(t) = \bar{r}(vt)$ . Unevenness with wavelength  $\lambda_x$  corresponds to harmonic excitation of the vehicle at frequency  $v/\lambda_x$ . The nonstationary ACF  $\phi_{r'r'}(t_1, t_2)$  of the stochastic excitation r'(t) is computed as follows:

$$\phi_{r'r'}(t_1, t_2) = \mathbb{E}[r'(t_1)r'(t_2)] = c'(t_1)c'(t_2)\phi_{\overline{r}'\overline{r}'}(t_2 - t_1)$$
(23)

The corresponding generalized PSD  $\Phi_{r'r'}(\omega_1, \omega_2) = \mathbb{E}[R'^*(\omega_1)R'(\omega_2)]$ , with \* denoting the complex conjugate, is found by a two-dimensional Fourier transform

$$\Phi_{r'r'}(\omega_1,\omega_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi_{r'r'}(t_1,t_2) \exp(i\omega_1 t_1 - i\omega_2 t_2) dt_1 dt_2$$
  
= 
$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} c'(t_1) c'(t_2) \phi_{\bar{r}'\bar{r}'}(t_2 - t_1)$$
  
× 
$$\exp(i\omega_1 t_1 - i\omega_2 t_2) dt_1 dt_2$$
(24)

An alternative expression for the generalized PSD  $\Phi_{r'r'}(\omega_1, \omega_2)$  is found by rewriting the ACF  $\phi_{\vec{r}'}(t_2 - t_1)$  of the underlying stationary stochastic process as the inverse Fourier transform of the PSD  $\Phi_{\vec{r}'\vec{r}'}(\omega')$ , finally leading to

$$\Phi_{r'r'}(\omega_1,\omega_2) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} C'^*(\omega_1 - \omega')C'(\omega_2 - \omega')\Phi_{\bar{r}'\bar{r}'}(\omega')d\omega'$$
(25)

where  $C'(\omega) =$  Fourier transform of c'(t). The PSD  $\Phi_{\vec{r}'\vec{r}'}(\omega)$  of the underlying stationary random process  $\vec{r}'(t)$  is computed from the PSD  $\Phi_{\vec{r}\vec{r}}(k_x)$  of the underlying homogeneous random field  $\bar{r}(x)$  in the wave number domain as follows (Schiehlen 2009):

$$\Phi_{\vec{r}'\vec{r}'}(\omega) = \frac{1}{\upsilon} \Phi_{\vec{r}\,\vec{r}}\left(\frac{\omega}{\upsilon}\right) \tag{26}$$

The PSD  $\Phi_{\vec{r}'\vec{r}'}(\omega')$  in Eq. (25) is now approximated as follows:

$$\Phi_{\vec{r}'\vec{r}'}(\omega') = \sum_{m=1}^{M} \Phi_{\vec{r}'\vec{r}'}(\omega'_m) \Delta \omega' \delta(\omega' - \omega'_m)$$
(27)

where  $\omega'_m = m\Delta\omega'$  and  $\Delta\omega'$  is the frequency spacing. The zero frequency component is not considered, because this corresponds to unevenness with an infinite wavelength and does not affect the dynamic vehicle response. Introducing Eq. (27) for the PSD  $\Phi_{\vec{r}\vec{r}'}(\omega')$  in Eq. (25) yields

$$\Phi_{r'r'}(\omega_1,\omega_2) = \frac{1}{2\pi} \sum_{m=1}^{M} C'^*(\omega_1 - \omega'_m) C'(\omega_2 - \omega'_m) \Phi_{r'r'}(\omega'_m) \Delta \omega'$$
(28)

#### Autocorrelation Function of the Interaction Force

The nonstationary ACF  $\phi_{ff}(t_1, t_2) = \mathbb{E}[f(t_1)f(t_2)]$  of the interaction force is now computed, based on the previously derived integral form, Eq. (14), of the compatibility equation in the deterministic case. First, the sum of the vehicle compliance and the bridge compliance  $c_v(t - \tau) + c_b(t, \tau)$  in Eq. (14) is replaced by  $c_{vb}(t, \tau)$ following the notation introduced in Eq. (15). Second, Eq. (14) is formulated for  $t = t_1$ ,  $\tau = \tau_1$  and  $t = t_2$ ,  $\tau = \tau_2$ , and the expected value of the product of both evaluations is computed. This leads to the following integral equation for the nonstationary ACF  $\phi_{ff}(t_1, t_2)$  of the interaction force:

$$\int_{-\infty}^{t_1} \int_{-\infty}^{t_2} c_{vb}(t_1, \tau_1) c_{vb}(t_2, \tau_2) \phi_{ff}(\tau_1, \tau_2) d\tau_1 d\tau_2$$
  
=  $u_{st}(t_1) u_{st}(t_2) + \phi_{t't'}(t_1, t_2)$  (29)

where use has been made of the fact that the mean value of r'(t) is zero. Eq. (29) is now formulated in the frequency domain as follows. First, a two-dimensional forward Fourier transform with respect to  $t_1$  and  $t_2$  is performed

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C_{vb}(-\omega_1, \tau_1) C_{vb}(\omega_2, \tau_2) \phi_{ff}(\tau_1, \tau_2) d\tau_1 d\tau_2 = U_{st}^*(\omega_1) U_{st}(\omega_2) + \Phi_{r'r'}(\omega_1, \omega_2)$$
(30)

Second, the ACF  $\phi_{ff}(\tau_1, \tau_2)$  of the interaction force f(t) is rewritten in terms of its two-dimensional Fourier transform

$$\phi_{ff}(\tau_1, \tau_2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi_{ff}(\omega'_1, \omega'_2) \exp(-i\omega'_1 \tau_1 + i\omega'_2 \tau_2) d\omega'_1 d\omega'_2$$
(31)

Substituting  $\phi_{ff}(\tau_1, \tau_2)$  from Eq. (31) into Eq. (30) and rearranging the terms allows recovery of the two-dimensional Fourier transform of the product  $C_{vb}(-\omega_1, \tau_1)C_{vb}(\omega_2, \tau_2)$ , with respect to  $\tau_1$  and  $\tau_2$ . Replacing  $\Phi_{r'r'}(\omega_1, \omega_2)$  on the right-hand-side of Eq. (30) by the approximation in Eq. (28) finally leads to the following integral equation for the generalized PSD  $\Phi_{ff}(\omega'_1, \omega'_2)$ :

$$\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C_{vb}(-\omega_1,\omega_1') C_{vb}(\omega_2,-\omega_2') \Phi_{ff}(\omega_1',\omega_2') d\omega_1' d\omega_2'$$
  
=  $U_{st}^*(\omega_1) U_{st}(\omega_2) + \frac{1}{2\pi} \sum_{m=1}^M C'^*(\omega_1 - \omega_m') C'(\omega_2 - \omega_m') \Phi_{\vec{r}'\vec{r}'}(\omega_m') \Delta \omega'$   
(32)

#### Autocorrelation Function of the Bridge Response

The nonstationary ACF  $\phi_{yy}(x_1, x_2, t_1, t_2) = \mathbb{E}[y(x_1, t_1)y(x_2, t_2)]$  of the bridge displacement is calculated based on the modal superposition in Eq. (3)

$$\phi_{yy}(x_1, x_2, t_1, t_2) = \mathbb{E}\left[\sum_{n=1}^{\infty} \phi_n(x_1) Z_n(t_1) \sum_{m=1}^{\infty} \phi_m(x_2) Z_m(t_2)\right]$$
$$= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_n(x_1) \phi_m(x_2) \phi_{z_n z_m}(t_1, t_2)$$
(33)

which shows that the ACF  $\phi_{yy}(x_1, x_2, t_1, t_2)$  is obtained from the cross-modal, cross-correlation functions (CCFs)  $\phi_{z_m z_n}(t_1, t_2)$ .

The following expression for the cross-modal CCFs  $\phi_{z_m z_n}(t_1, t_2)$ in terms of the generalized PSD  $\Phi_{ff}(\omega_1, \omega_2)$  is derived using Eq. (10)

$$\begin{aligned} \phi_{z_n z_m}(t_1, t_2) &= \mathbb{E}[z_n(t_1) z_m(t_2)] \\ &= \mathbb{E}[z_n(t_1 z_m^*(t_2)] = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_n(t_1, \omega_1) g_m^*(t_2, \omega_2) \\ &\times \Phi_{ff}(\omega_1, \omega_2) d\omega_1 \omega_2 \end{aligned}$$
(34)

When the generalized PSD  $\Phi_{ff}(\omega_1, \omega_2)$  of the variable moving load f(t) has been obtained, Eq. (34) can be used to compute the cross-modal CCFs  $\phi_{z_m z_n}(t_1, t_2)$  and, subsequently, by Eq. (33), the nonstationary ACF of the bridge response.

## Numerical Solution

#### Deterministic Case

The integral Eq. (17) for the variable moving load is solved by assuming a constant value of  $F(\omega)$  in each frequency interval  $[\omega_i - 0.5\Delta\omega, \omega_i + 0.5\Delta\omega]$  of width  $\Delta\omega$  centered at  $\omega_i = j\Delta\omega$ 

$$\hat{F}(\omega) = \sum_{j=1}^{N_{\omega}} \hat{F}_j \Delta(\omega - \omega_j)$$
(35)

where the hat denotes the approximation and the function  $\Delta(\omega - \omega_j) = 1$  for  $\omega \in [\omega_j - 0.5\Delta\omega, \omega_j + 0.5\Delta\omega]$  and zero elsewhere. The interval centered at  $\omega_j = 0$  is not considered, because

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the variable moving load f(t) has a zero mean value and, therefore,  $F(\omega)$  is zero for  $\omega = 0$ .

A system of  $N_{\omega}$  equations with  $N_{\omega}$  unknowns  $\hat{F}_j$  is obtained by formulating the integral Eq. (17) for  $N_{\omega}$  frequencies  $\omega' = \omega'_k = k \Delta \omega$ 

$$C_{v}(\omega_{k}')\hat{F}_{k} + \sum_{j=1}^{N_{\omega}} \left[ \frac{1}{2\pi} \int_{\omega_{j}-0.5\Delta\omega}^{\omega_{j}+0.5\Delta\omega} C_{b}(\omega_{k}',-\omega)d\omega \right] \hat{F}_{j}$$
$$= -U_{st}(\omega_{k}') - R'(\omega_{k}')$$
(36)

When  $\Delta \omega$  is sufficiently small, the second term can be approximated as follows:

$$C_{\upsilon}(\omega_k')\hat{F}_k + \sum_{j=1}^{N_{\omega}} \left[ \frac{1}{2\pi} C_b(\omega_k', -\omega_j) \Delta \omega \right] \hat{F}_j = -U_{st}(\omega_k') - R'(\omega_k')$$
(37)

The system of equations is rewritten in the following matrix form:

$$(\mathbf{C}_v + \mathbf{C}_b)\hat{\mathbf{F}} = -\mathbf{U}_{st} - \mathbf{R}' \tag{38}$$

where  $\mathbf{C}_v$  is the  $N_\omega \times N_\omega$  diagonal matrix that represents the vehicle compliance,  $\mathbf{C}_b$  is the  $N_\omega \times N_\omega$  bridge compliance matrix,  $\hat{\mathbf{F}}$  is the  $N_\omega \times 1$  vector that collects the unknowns  $\hat{F}_j (j = 1, ..., N_\omega)$ , and  $\mathbf{U}_{st}$ and  $\mathbf{R}'$  are the  $N_\omega \times 1$  vectors that collect the displacement at the contact point  $U_{st}(\omega)$  because of the constant moving load  $f_{st}$  and the unevenness  $R'(\omega)$ , respectively. Eq. (38) shows that the interaction force results from two excitation mechanisms. First, in the case of a perfectly smooth road surface, the deflection of the bridge at the contact point will lead to a dynamic excitation of the vehicle. This effect is represented by the first term  $U_{st}(\omega)$  on the right-handside of Eq. (38). Second, unevenness of the bridge roadway surface, represented by the second term  $\mathbf{R}'$ , will lead to additional dynamic excitation, which will dominate in the case of a very stiff bridge, i.e., when  $U_{st}(\omega)$  is small compared with  $\mathbf{R}'$ .

An inversion of the sum of the vehicle compliance matrix  $C_v$  and the bridge compliance matrix  $C_b$  allows rewriting of the system as follows:

$$\hat{\mathbf{F}} = -(\mathbf{C}_v + \mathbf{C}_b)^{-1}(\mathbf{U}_{st} + \mathbf{R}') = \mathbf{H}_{fu}(\mathbf{U}_{st} + \mathbf{R}')$$
(39)

where  $\mathbf{H}_{fu}$  is the  $N_{\omega} \times N_{\omega}$  transfer matrix of the coupled vehiclebridge system that relates the Fourier transform of the variable moving load  $\hat{\mathbf{F}}$  to the combined excitation  $\mathbf{U}_{st} + \mathbf{R}'$ .

The modal response  $z_n(t)$  is subsequently calculated by substituting Eq. (35) in Eq. (10)

$$z_n(t) = \sum_{j=1}^{N_\omega} \left[ \frac{1}{2\pi} \int_{\omega_j - 0.5\Delta\omega}^{\omega_j + 0.5\Delta\omega} g_n(t, -\omega) d\omega \right] \hat{F}_j$$
(40)

When  $\Delta \omega$  is sufficiently small, the bracketed term can be approximated by  $1/2\pi g_n(t, -\omega_j)\Delta \omega$  or, equivalently,  $1/2\pi g_n^*(t, \omega_j)\Delta \omega$ . When Eq. (40) is evaluated for  $N_t$  time steps  $t_k = k\Delta t$ , it can be rewritten in the following matrix form:

$$\hat{\mathbf{z}}_n = \hat{\mathbf{G}}_n^* \hat{\mathbf{F}} \tag{41}$$

where the  $N_t \times 1$  vector  $\hat{\mathbf{Z}}_n$  collects the sequence  $z_n(t_k)$ ,  $k = 1, ..., N_t$  and the  $N_t \times N_\omega$  matrix  $\hat{\mathbf{G}}_n^*$  contains the bracketed terms in Eq. (40). The bridge displacement  $y(x, t_k)$  is obtained by the modal superposition in Eq. (3).

#### Stochastic Case

## Autocorrelation Function of the Interaction Force

Instead of computing the generalized PSD  $\Phi_{ff}(\omega'_1, \omega'_2)$  directly from Eq. (32), an expression for  $\Phi_{ff}(\omega'_1, \omega'_2)$  is derived from the numerical solution in Eq. (39) for the deterministic case

$$\hat{\Phi}_{ff} = \mathbb{E}\{\hat{\mathbf{F}}\,\hat{\mathbf{F}}^H\} = \mathbb{E}\{[\mathbf{H}_{fu}^*(\mathbf{U}_{st}^* + \mathbf{R}'^*)] \otimes [\mathbf{H}_{fu}(\mathbf{U}_{st} + \mathbf{R}')]\} \quad (42)$$

where H = Hermitian or conjugate transpose and  $\otimes$  = vector or outer product. The  $N_{\omega} \times N_{\omega}$  matrix  $\hat{\Phi}_{ff}$  represents the generalized PSD  $\Phi_{ff}(\omega'_1, \omega'_2)$  at discretized frequencies  $\omega'_1, \omega'_2 = \omega'_k = k\Delta\omega$ ,  $k = 1, ..., N_{\omega}$ . Because the transfer matrix  $\mathbf{H}_{fu}$  and the vector  $\mathbf{U}_{st}$  are deterministic quantities, the right-hand-side of Eq. (42) becomes

$$\hat{\Phi}_{ff} = \mathbf{H}_{fu}^* \mathbf{U}_{st}^* \mathbf{U}_{st}^T \mathbf{H}_{fu}^T + \mathbf{H}_{fu}^* \mathbb{E}[\mathbf{R}'^* \mathbf{R}'^T] \mathbf{H}_{fu}^T$$
(43)

where use has been made of the fact that  $\mathbb{E}[\mathbf{R}'] = 0$ . The term  $\mathbb{E}[\mathbf{R}'^*\mathbf{R}'^T]$  matrix  $\hat{\Phi}_{r'r'}$ , which represents the generalized PSD  $\hat{\Phi}_{r'r'}(\omega'_1, \omega'_2)$  at discretized frequencies  $\omega'_1, \omega'_2 = \omega'_k = k\Delta\omega$ ,  $k = 1, ..., N_{\omega}$ . Introducing the approximation in Eq. (28) for the generalized PSD  $\Phi_{r'r'}$  in Eq. (43) leads to

$$\Phi_{ff} = \mathbf{H}_{fu}^* \mathbf{U}_{st}^* \mathbf{U}_{st}^T \mathbf{H}_{fu}^T + \frac{1}{2\pi} \sum_{m=1}^M [\mathbf{H}_{fu}^* \mathbf{C}^*(\omega_m') \mathbf{C}^T(\omega_m') \mathbf{H}_{fu}^T] \Phi_{\vec{r}' \vec{r}'}(\omega_m') \Delta \omega' \qquad (44)$$

where the  $N_{\omega} \times 1$  vector  $\mathbf{C}^*(\omega'_m)$  collects the terms  $C'^*(\omega'_k - \omega'_m), \omega'_k = k\Delta\omega, \ k = 1, \dots, N_{\omega}$  for a particular value of  $\omega'_m$ .

The first term on the left-hand-side of Eq. (44) represents the contribution of the deterministic excitation  $\mathbf{U}_{st}$  to the generalized PSD  $\hat{\Phi}_{ff}$  and can be computed by solving the system of Eq. (39) in the deterministic case with  $\mathbf{R}' = 0$ . The second term can be computed by solving Eq. (39) as well for each of the M excitations  $\mathbf{R}' = (1/\sqrt{2\pi})\mathbf{C}(\omega'_m)\sqrt{\Phi_{\vec{r}'\vec{r}}(\omega'_m)\Delta\omega'}$ , representing a modulated harmonic excitation with frequency  $\omega'_m$ . In this way, an efficient solution of the stochastic problem with random road unvenenness is obtained. A similar solution strategy is followed in the pseudoexcitation method (Zhang et al. 2010).

#### Autocorrelation Function of the Bridge Response

Once the generalized PSD  $\tilde{\Phi}_{ff}$  of the interaction force has been obtained, the nonstationary ACF of the bridge response can be computed. This requires the calculation of the cross-modal CCFs  $\phi_{z_n \bar{z}_m}(t_1, t_2)$ . Based on Eq. (41) for the modal coordinates  $z_n(t)$  in the deterministic case, the following expression is obtained:

$$\hat{\phi}_{z_n z_m} = \mathbb{E}[\hat{\mathbf{G}}_n \hat{\mathbf{F}}^* \otimes \hat{\mathbf{G}}_m^* \hat{\mathbf{F}}] = \hat{\mathbf{G}}_n \hat{\Phi}_{ff} \hat{\mathbf{G}}_m^{*T}$$
(45)

where  $\phi_{z_n z_m}$  is the  $N_t \times N_t$  matrix with the discretized approximation of the cross-modal CCF  $\Phi_{z_n z_m}(t_1, t_2)$ . Next, the numerical solution in Eq. (44) for the generalized PSD  $\hat{\Phi}_{ff}$  is inserted in Eq. (45)

$$\hat{\phi}_{z_n z_m} = \hat{\mathbf{G}}_n \mathbf{H}_{fu}^* \mathbf{U}_{st}^* \mathbf{U}_{st}^T \mathbf{H}_{fu}^T \hat{\mathbf{G}}_m^{*T} + \frac{1}{2\pi} \sum_{m=1}^M [\hat{\mathbf{G}}_n \mathbf{H}_{fu}^* \mathbf{C}^*(\omega'_m) \mathbf{C}^T(\omega'_m) \mathbf{H}_{fu}^T \hat{\mathbf{G}}_m^{*T}] \Phi_{\vec{r}' \vec{r}'}(\omega_m) \Delta \omega'$$
(46)

As in the case of Eq. (44), the solution is obtained by solving one deterministic problem where the deterministic excitation  $\mathbf{U}_{st}$  is considered and M additional deterministic problems each with an excitation  $\mathbf{R}' = (1/\sqrt{2\pi})\mathbf{C}(\omega'_m)\sqrt{\Phi_{\vec{r}'\vec{r}'}(\omega'_m)\Delta\omega'}$  (m = 1, ..., M).

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The  $N_t \times N_{\omega}$  matrices  $\hat{\mathbf{G}}_n(n = 1, ..., N)$  and the  $N_{\omega} \times N_{\omega}$  transfer matrix  $\mathbf{H}_{fu}$  only need to be computed once. As a result, the nonstationary second-order statistical characteristics of the response of the vehicle-bridge system are obtained at a moderate computational cost.

## **Application Examples**

In this subsection, the passage of a heavy vehicle on a single span highway bridge is considered. The vehicle is a 40-ton truck that is modeled as a quarter-car model with 2 degrees of freedom (DOFs) (Cebon 1999). The constant load  $f_{st}$  of the vehicle is equal to -392.4 kN. The sprung mass of the vehicle  $m_s = 36,000$  kg is supported by the suspension system that is represented by a spring-dashpot connection with a spring stiffness  $k_s = 18$  MN/m and a dashpot constant  $c_s = 1.4$  MNs/m. The suspension system is connected to the unsprung mass of the vehicle  $m_u = 4,000$  kg, which is supported by the tires that are represented by a spring-dashpot connection with a spring stiffness  $k_t =$ 72 MN/m and a dashpot constant  $c_t = 1.4$  MNs/m. The two DOFs are the vertical displacement of the sprung and unsprung masses, respectively. The vehicle has two vibration modes, a bouncing mode of the sprung mass with a natural frequency of 3.18 Hz and an axle hop mode with a natural frequency of 23.9 Hz.

The bridge model is a simply supported beam model for the Pirton Lane Highway bridge in Gloucester (UK) (Cebon 1999). The bridge has a length L = 40 m, an estimated mass per unit length of  $\bar{m} = 12,000$  kg/m, a first natural frequency  $\omega_1 = 2\pi \times$ 3.20 rad/s, and modal damping ratio  $\xi_1 = 0.02$ . The first natural frequencies of the vehicle and the bridge (considered individually) are similar, and therefore the dynamic vehicle-bridge interaction is expected to have an important effect on the dynamic response of the bridge (Cantieni 1992). Based on the value for  $\omega_1$ , the bending stiffness EI is estimated as  $EI = \bar{m}L^4\omega_1^2/\pi^4 = 1.26 \times 10^5 \text{ mN} - \text{m}^2$ . The modal damping ratio  $\xi_1$  is used to estimate the viscous damping in the beam. Only the viscous resistance to the vertical velocity of the beam c in Eq. (1) is considered, because this allows verification of the solution for a moving load with constant intensity to the closed-form solution given by Frýba (1996). The damping constant c is computed from the modal parameters  $\omega_1$  and  $\xi_1$  as  $c = 2\bar{m}\omega_1\xi_1 = 9.60$  kN – s/m. In the solution by modal superposition, the first three bridge modes are considered with natural frequencies  $\omega_1 = 20 \text{ rad/s}$ ,  $\omega_2 = 80 \text{ rad/s}$ , and  $\omega_3 = 180 \text{ rad/s}$ , respectively. The corresponding modal damping ratios are equal to  $\xi_1 = 0.02$ ,  $\xi_2 = 0.005$ , and  $\xi_3 = 0.002$ , respectively.

The proposed solution procedure is used to compute the dynamic response of the bridge to the passage of the 40-ton vehicle at a speed of v = 100 km/h. First, the case of a perfectly smooth road surface is considered. Next, the case is considered where the road surface unevenness is irregular and can be represented by a random field.

# Vehicle-Bridge Interaction for a Perfectly Smooth Road Surface

In the case where the road surface is perfectly smooth, the coupled vehicle-bridge system is only excited by the first term on the right-hand-side of Eqs. (17) and (39), which represents the Fourier transform  $U_{st}(\omega)$  of the displacement of the bridge at the contact point x = vt because of the constant moving load  $f_{st}$ .

Fig. 2(b) shows the Fourier amplitude spectrum  $|U_{st}(\omega)|$  of the displacement at the contact point. The Fourier amplitude spectrum is concentrated at low frequencies  $f \leq 5$  Hz, revealing the



**Fig. 2.** (a) Time history  $y_{st}(t)$ ; (b) Fourier amplitude spectrum  $|U_{st}(\omega)|$  of the displacement at the contact point to a constant moving load  $f_{st} = -392.4$  kN. The vertical dotted lines indicate the times t = 0 and  $t = t_d$  at which the vehicle enters and leaves the bridge, respectively

low-frequency character of the excitation in Eq. (17). The time history  $u_{st}(t)$  [Fig. 2(a)] of the response has been obtained by means of an inverse Fourier transform of  $U_{st}(\omega)$ .

The intensity of the variable moving load is now computed from the solution of the system of Eq. (39), based on the vehicle compliance, the bridge compliance, and the excitation  $U_{st}(\omega)$ . The Fourier amplitude spectrum of the variable moving load  $|F(\omega)|$ [Fig. 3(b)] shows a high spectral peak near 3.2 Hz, where the first natural frequency of the bridge and the vehicle are found and a second peak near the second natural frequency of the bridge at 12.7 Hz is found. The corresponding time history of the interaction force f(t) is shown in Fig. 3(a). Because no unevenness is present on the road, the vehicle is not excited prior to its entrance on the bridge and the intensity of the variable moving load f(t) = 0 for  $t \leq 0$ . After excitation by the bridge, the force f(t) shows a nearly harmonic behavior with a period corresponding to the first natural frequency of the bridge and the vehicle. The peak value of the variable moving load is 6.1 kN, corresponding to only 1.6% of the constant load  $f_{st} = -392.4$  kN. After the vehicle leaves the bridge  $(t_d \le t)$ , the force is no longer applied to the bridge, but to the rigid, smooth pavement on which the vehicle travels after its passage on the bridge.

Based on the Fourier transform  $F(\omega)$  of the variable moving load, the total response of the bridge is obtained by introducing the total load  $f_{st}2\pi\delta(\omega) + F(\omega)$  in Eq. (10) for the time history of the modal coordinates and applying the modal superposition in Eq. (3). Fig. 4(b) compares the Fourier amplitude spectrum of the bridge displacement at midspan because of the total load  $f_{st}2\pi\delta(\omega) + F(\omega)$  with the displacement because of a moving load with constant amplitude  $f_{st}$ . At low frequencies, the results are similar, whereas around the first natural frequency of the vehicle and the bridge, a small difference is found because of the additional



**Fig. 3.** (a) Time history f(t); (b) Fourier amplitude spectrum  $|F(\omega)|$  of the variable moving load. The vertical dotted lines indicate the times t = 0 and  $t = t_d$  at which the vehicle enters and leaves the bridge, respectively

variable moving load  $F(\omega)$ . In Fig. 4(a), the time history of the displacement at midspan because of  $f_{st}$  is compared with the corresponding closed-form solution of Frýba (1996) for the forced vibration phase  $0 \le t \le t_d$ . An excellent agreement is found



**Fig. 4.** (a) Time history; (b) Fourier amplitude spectrum of the bridge displacement at midspan because of the total moving load  $f_{st} + f(t)$  (solid line) and the constant moving load  $f_{st}$  (dotted line). The vertical dotted lines indicate the times t = 0 and  $t = t_d$  at which the vehicle enters and leaves the bridge, respectively. The circles on (a) correspond to the closed form solution for the response because of  $f_{st}$  in the forced vibration phase  $0 \le t \le t_d$ 

between the closed-form solution and the obtained solution for the displacement at midspan because of  $f_{st}$ .

## Vehicle-Bridge Interaction Because of Random Road Unevenness

In this section, the passage of the 40-ton vehicle on the same bridge, but with random road unevenness, is considered. The nonstationary ACF of the vertical bridge displacement  $\phi_{yy}(x_1, x_2, t_1, t_2)$  is computed for the case in which the underlying homogeneous random field  $\bar{r}(x)$  assumes the PSD (Braun and Hellenbroich 1991; ISO 1991)

$$\Phi_{\bar{r}\bar{r}}(k_x) = \Phi_{\bar{r}\bar{r}}(k_{x0}) \left(\frac{k_x}{k_{x0}}\right)^{-w}$$
(47)

where w = 2,  $k_x 0 = 1$  rad/m, and the reference value  $\Phi_{\bar{r}\bar{r}}(k_{x0}) =$  $2\pi \times 10^{-6}$  m<sup>3</sup>/rad corresponding to the average of class A road unevenness described in ISO8608 (ISO 1991). The additional factor  $2\pi$  in the value for  $\Phi_{\bar{r}\bar{r}}(k_{x0})$  is because of the assumed convention in Eq. (8) for the Fourier transform pair. The PSD in Eq. (47) cannot be used in the entire wave number range  $(0-\infty)$ , because it leads to infinite values for the limiting case  $k_x \rightarrow 0$  (Schiehlen 2009). The PSD is, therefore, truncated at a lower limit of  $k_x =$  $2\pi \times 0.04$  rad/m and an upper limit of  $k_x = 2\pi \times 0.20$  rad/m, so that unevenness in a range of wavelengths between 5 and 25 m is considered. For a vehicle speed of v = 100 km/h, this results in excitation in the frequency range between 1.11 and 5.56 Hz that contains the first natural frequency of the bridge and the vehicle. The corresponding PSD  $\Phi_{\vec{r}'\vec{r}'}(\omega')$  in terms of the circular frequency  $\omega'$  is found according to Eq. (26). This PSD is approximated according to Eq. (27) with  $\Delta \omega' = v \Delta k_x$  and  $\Delta k_x =$  $2\pi \times 5 \times 10^{-4}$  rad/m, so that a total of M = 321 terms is considered in the Eq. (27). For a vehicle speed  $v = 100 \text{ km/h}, \Delta \omega' =$  $2\pi \times 0.014$  rad/s.

The nonstationary random process  $r'(t) = c'(t)\overline{r}'(t)$  is obtained by considering the following Gaussian modulation function c(x), see Eq. (21):

$$c(x) = \exp[-a(x - L/2)^2]$$
(48)

where *a* is taken equal to  $1/(2L)^2$ , and therefore the road unevenness  $r(x) = c(x)\overline{r}(x)$  on the bridge is close to the underlying stationary random field  $\overline{r}(x)$ .

The corresponding generalized PSD  $\Phi_{r'r'}(\omega_1, \omega_2)$  from Eq. (28) becomes

$$\Phi_{r'r'}(\omega_1,\omega_2) = \frac{1}{2\pi} \sum_{m=1}^M C'^*(\omega_1 - \omega'_m)C'(\omega_2 - \omega'_m)\Phi_{\bar{r}\bar{r}}(k_{x0})v \left(\frac{k_{x0}}{\omega'_m}\right)^2 \Delta\omega'$$
(49)

where  $C'(\omega) =$  Fourier transform of c'(t) = c(vt). Terms as  $C'^*(\omega_1 - \omega'_m)$  in Eq. (49) are computed as the Fourier transform of  $c'(t) \exp(i\omega'_m t)$ . Based on this expression, the nonstationary statistical characteristics of the interaction force f(t) and the vertical bridge displacement y(x, t) are computed. To verify the results, realizations of the random process r'(t) are generated according to the spectral representation theorem (Shinozuka and Deodatis 1991; Shinozuka and Jan 1972) as a superposition of harmonic functions with random phase angles

$$r'(t) = c(t) \sum_{m=1}^{M} \frac{1}{\sqrt{2\pi}} \sqrt{2\Phi_{\vec{r}'\vec{r}'}(\omega_m)\Delta\omega} \cos(\omega_m t - \theta_m)$$
(50)

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where  $\theta_m$  = independent random phase angles uniformly distributed in the interval (0,  $2\pi$ ). The samples have a period  $T = 2\pi/\Delta\omega' =$ 72 s and are asymptotically Gaussian as M tends to infinity and  $\Delta\omega'$  tends to zero for a fixed value of  $\omega_m^{max} = M\Delta\omega'$ . Fig. 5(a) shows two realizations of the random process r'(t) according to Eq. (50).

To verify Eq. (49) for the generalized PSD  $\Phi_{r'r'}(\omega_1, \omega_2)$ , the time-dependent mean square value  $\sigma_{r'}^2(t)$  of the random process r'(t) is computed from the two-dimensional inverse Fourier transform  $\phi(t_1, t_2)r'r'$  of the generalized PSD  $\Phi_{r'r'}(\omega_1, \omega_2)$ . Fig. 5(b) shows that a good agreement is obtained between the computed value of  $\sigma_r'(t)$  and the value that has been estimated from 2,048 realizations. This relatively high number of realizations was required to obtain convergence of the statistics of the bridge response shown in the subsequent paragraphs.

Next, the intensity of the variable moving load f(t) is computed. In the calculation, only the excitation  $R'(\omega)$ , because of the unevenness r'(t), is taken into account on the right-hand-side of Eq. (17) in the deterministic case and Eq. (32) in the stochastic case. The first term  $U_{st}(\omega)$  on the right-hand-side of these equations has already been considered in the previous subsection. The total response is, therefore, found by superposing the solution previously presented to all results shown next. Fig. 5(a) shows the deterministic solution of the variable moving load f(t) for the two samples of r'(t). Outside the time range  $0 \le t \le t_d$ , the vehicle is not on the bridge and the variable moving load f(t) is applied on the rigid road pavement assumed in front of the bridge  $(t \le 0)$  and behind the bridge  $(t_d \le t)$ .

The generalized PSD  $\Phi_{ff}(\omega_1, \omega_2)$  of the interaction force is now computed according to Eq. (44), where only the second term on the right-hand-side is considered. Fig. 6(b) shows that a good agreement is obtained between the value of  $\sigma_f(t)$  obtained from the





**Fig. 6.** (a) Time history of the variable moving load f(t) for two realizations of r'(t); (b) comparison of the standard deviation  $\sigma_f(t)$  (black) and the estimated value based on 2,048 realizations (gray). The vertical dotted lines indicate the times t = 0 and  $t = t_d$  at which the vehicle enters and leaves the bridge, respectively



**Fig. 5.** (a) Two realizations of the random process r'(t); (b) comparison of the standard deviation  $\sigma'_r(t)$  (black) and the estimated value based on 2,048 realizations (gray). The vertical dotted lines indicate the times t = 0 and  $t = t_d$  at which the vehicle enters and leaves the bridge, respectively

**Fig. 7.** (a) Time history of the vertical bridge displacement at midspan y(t) for two realizations of r'(t); (b) comparison of the computed standard deviation  $\sigma_y(t)$  (black) and the estimated value based on 2,048 realizations (gray). The vertical dotted lines indicate the times t = 0 and  $t = t_d$  at which the vehicle enters and leaves the bridge, respectively

two-dimensional inverse Fourier transform  $\phi_{ff}(t_1, t_2)$  of the generalized PSD  $\Phi_{ff}(\omega_1, \omega_2)$  and the estimated value based on 2,048 realizations. Furthermore, the results for  $\sigma_f(t)$  suggest that in the present case, vehicle-bridge interaction results in a reduction of the vehicle load. Comparing the standard deviation  $\sigma_f(t)$  in Fig. 6(b) with the time history of the force in Fig. 6(a) shows that the variable moving load f(t) because of road unevenness is much larger than for the excitation resulting from the bridge deflection because of the constant moving load  $f_{st}$ . The standard deviation of the variable moving load  $\sigma_f(t)$  reaches a value of 27.4 kN, i.e., 7.0% of the constant load  $f_{st} = -392.4$  kN.

Finally, the time history of the vertical displacement of the bridge at midspan is computed. Fig. 7(a) shows the deterministic solution of the time history y(x, t) of the displacement at midspan x = L/2 for the samples of r'(t). The displacement increases as the vehicle enters the bridge at t = 0 and reduces again as the vehicle leaves the bridge. At times  $t > t_d$ , the bridge is no longer excited by the vehicle and is in decaying free vibration.

The cross-modal CCFs  $\phi_{z_m z_n}(t_1, t_2)$  are computed from the PSD  $\Phi_{\vec{r}'\vec{r}'}(\omega')$  by means of Eq. (46), where only the second term on the right-hand-side is considered. The nonstationary ACF  $\phi_{yy}(x_1, x_2, t_1, t_2)$  of the bridge displacements at locations  $x_1$  and  $x_2$  is evaluated subsequently using Eq. (33). Fig. 7(b) shows that a good agreement is obtained between the computed value of  $\sigma_y(t) = \sqrt{\phi(L/2, L/2, t, t)}$  and the estimated value based on 2,048 realizations, thus validating the proposed analytical solution procedure.

## Conclusion

The vehicle-bridge interaction problem is solved by means of a compliance formulation in a frame of reference that moves with the vehicle. An expression is derived for the variable moving load that shows how the Fourier transform of the force is determined from the bridge displacement at the contact point for the constant component of the vehicle load and the road unevenness. This provides a clear physical insight into the importance of both excitation mechanisms for vehicle-bridge interaction. Furthermore, an efficient solution procedure is presented for the nonstationary second-order statistical characteristics of the bridge response in the case where the unevenness is modeled as a random field. The solution procedure is illustrated by an example where the passage of a heavy vehicle on a single span highway bridge is considered. For the calculations, use is made of closed-form solutions that are available for a simply supported Euler-Bernoulli beam model for the bridge. The solution procedure for the second-order statistical characteristics of the bridge response is successfully validated by means of Monte Carlo simulations.

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