ELSEVIER

0266-8920(95)00026-7

Probabilistic Engineering Mechanics 11 (1996) 37–50

0266-8920/96/$15.00

An explicit closed-form solution for linear systems subjected to nonstationary random excitation

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Explicit, closed-form solutions are presented for the correlation matrix and evolutionary power spectral density matrix of the response of a linear, classically damped MDOF system subjected to a uniformly modulated random process with the gamma envelope function. The effects of the statistical cross-modal correlations on the evolutionary mean square responses are investigated. A simple MDOF system subjected to ground motion is used as an illustrative example. Through this study, additional insight is gained into the nonstationary behavior of linear dynamic systems.

INTRODUCTION

The nonstationary stochastic response of linear systems to nonstationary random excitations is of great interest in engineering fields, such as structural earthquake engineering and aerospace engineering. However, closed-form solutions for the statistical moments of the nonstationary response to a generally defined nonstationary input exist only in integral form. Only a few explicit closed-form solutions (in terms of elementary functions) exist for particular cases of nonstationary inputs. Therefore, a number of approximate closed-form solutions for certain types of nonstationary excitation and numerical integration schemes have been developed by various researchers. The most common type of nonstationary input for which analytical solutions (exact or approximate) have been proposed is the uniformly (amplitude) modulated or separable random process which is defined as the product of a stationary process and a deterministic envelope function, also called the time modulating function. In the literature, the square of the envelope function is referred to as the strength function. The envelope or strength functions considered in the past include the step,[1] boxcar,[2] staircase,[3] half sine,[4] periodic,[5] exponential,[6] gamma,[7] beta,[8] and piecewise linear functions.[9–12] The mean square response and mean energy response of a linear SDOF structure subjected to various evolutionary random processes was examined by Spanos[13] using approximate analytical expressions.

Exact closed-form solutions in terms of elementary functions exist for the following cases. Caughey and Stumpf[1] first analyzed the transient mean square response of a linear single-degree-of-freedom (SDOF) oscillator to unit step modulated white noise. Barnoski and Maurer[2] examined the mean square response of a linear SDOF system excited by both white noise and noise with an exponentially decaying harmonic correlation function for both the unit step and boxcar time modulating functions. Corotis and Vanmarcke[14] studied the evolutionary (time-dependent) power spectrum of the response of a linear SDOF system exposed to a unit-step modulated stationary process. The evolutionary power spectrum of the response of a SDOF oscillator subjected to an exponentially modulated stationary process was also investigated by Corotis and Marshall.[15] The evolutionary response covariance matrices of a multi-degree-of-freedom (MDOF) system subjected to a piecewise linear modulated white noise process were derived by Gasparini and DebChaudhury[9,10] via the state-space approach. To[16] presented explicit closed-form expressions for the evolutionary cross-spectral density matrix of the responses of a MDOF system subjected to exponentially decaying random excitations. Iwan and Hou[17] provided explicit solutions for the second-moment statistics of the response of SDOF systems excited by a white noise process modulated with the unit step, boxcar and gamma envelope functions. Grigoriu[18] developed a new method, based on the properties of conditional Gaussian variables and the system transition matrix, to obtain the transient first- and second-order response statistics of a time-invariant stable linear system subjected to a stationary Gaussian input.

This paper presents general analytical expressions for the correlation matrix and evolutionary power spectral density matrix of the response of a linear, classically damped, MDOF system subjected to a general evolutionary vector process defined in terms of time-frequency modulating functions. Some of this work has been developed earlier by Priestley in his book on multivariate time series.[19] Here, the solutions are obtained within the framework of the modal superposition approach. The explicit closed-form solution for the evolutionary power spectral density matrix of the response of a MDOF system subjected to a uniformly modulated random process with the gamma envelope function is derived. From this solution, the response correlation matrix of the MDOF system can be obtained using numerical integration. However, the explicit closed-form solution for the response correlation matrix is derived for the case of gamma modulated white noise excitation. It is noted that the solution for the correlation matrix between the responses at two different times ($t_1 = t$ and $t_2 = t + \tau$) is given.

Using the analytical solutions derived in the paper, the effects of the statistical cross-modal correlations on the evolutionary mean square responses are investigated. A simple application example considering ground motion excitation is used to illustrate the findings of the paper.

FORMULATION

System equations

Consider the matrix equation of motion of a linear MDOF system

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{P}\mathbf{F}(t) \quad (1)$$

where **M**, **C**, and **K** are the $n \times n$ time-invariant mass, damping and stiffness matrices, respectively; $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$, and $\ddot{\mathbf{U}}(t)$ are the length-n vectors of nodal displacements, velocities and accelerations, respectively; **P** is the $n \times m$ load distribution matrix; and $\mathbf{F}(t)$ is the length-m vector of external load functions which, in the case of random excitations, is a random vector process. It is noted that equation (1) also applies to a system excited by the motion of its support points. In this case, $\mathbf{P} = -\mathbf{ML}$, where column j of the $n \times m$ influence coefficient matrix **L** represents the pseudo-static response in all degrees of freedom due to a unit support motion (translational or rotational) in direction j; $\mathbf{F}(t)$ is the base acceleration vector and $\mathbf{U}(t)$ represents the vector of nodal displacements relative to the rigid base of the structure. Assuming $\mathbf{X}(t)$ to be the length-n vector of absolute nodal displacements, equation (1) can be rewritten as

$$\mathbf{M}\ddot{\mathbf{X}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{0}. \quad (2)$$

External force vector

In the frequency domain there are several approaches to describe theoretically nonstationary processes including the generalized (double-frequency) spectrum,[20] the instantaneous (frequency-time) spectrum,[21] the evolutionary spectrum,[22–23] and the physical spectrum.[24] The present case uses Priestley's evolutionary process model, which has a particularly convenient input–output relationship for linear systems.[25] Priestley's definition and characterization of an evolutionary scalar process can be readily extended to the case of a vector process. Thus, an evolutionary, zero-mean vector process $\mathbf{F}(t)$ can be expressed in Fourier–Stieltjes integral form as

$$\mathbf{F}(t) = \int_{-\infty}^{\infty} \mathbf{A}_{\mathbf{F}}(\omega, t)e^{\mathrm{j}\omega t}\,\mathrm{d}\mathbf{Z}(\omega) \quad (3)$$

in which $\mathrm{j} = \sqrt{-1}$, ω denotes the circular frequency, $\mathbf{F}(t)$ is a length-m vector

$$\mathbf{A}_{\mathbf{F}}(\omega, t) = \begin{bmatrix} \mathbf{a}_{F_1}(\omega, t) \\ \mathbf{a}_{F_2}(\omega, t) \\ \cdots \\ \mathbf{a}_{F_m}(\omega, t) \end{bmatrix}_{(m \times k)} \quad (4)$$

is a matrix of frequency-time modulating functions of $\mathbf{F}(t)$, $\mathbf{a}_{F_j}(\omega, t)$ is a length-k row vector of frequency-time modulating functions of $F_j(t)$, and $\mathrm{d}\mathbf{Z}(\omega)$ is a length-k orthogonal-increment vector process having the properties:

$$E[\mathrm{d}\mathbf{Z}(\omega)] = \mathbf{0} \quad (5)$$

and

$$E[\mathrm{d}\mathbf{Z}^*(\omega_1)\mathrm{d}\mathbf{Z}^{\mathrm{T}}(\omega_2)] = \mathbf{\Phi}(\omega_1)\delta(\omega_1 - \omega_2)\mathrm{d}\omega_1\,\mathrm{d}\omega_2 \quad (6)$$

in which $E[\ldots]$ denotes the expectation or ensemble-average operator. In equation (6), $\mathbf{\Phi}(\omega)$ is a $k \times k$ Hermitian matrix whose diagonal terms are non-negative even functions, $\delta(\ldots)$ represents the Dirac delta function and the superposed T and * denote the transpose and the complex conjugate, respectively, of the corresponding variable or function. Hammond[26] considers the case in which $\mathbf{A}_{\mathbf{F}}(\omega, t)$ is a diagonal matrix and, therefore, each component of $\mathbf{F}(t)$ is a so-called oscillatory process. If the components of $\mathrm{d}\mathbf{Z}(\omega)$ are mutually independent, then each component of $\mathbf{F}(t)$ is defined by Battaglia[27] as a sigma-oscillatory process. Notice that if $\mathbf{A}_{\mathbf{F}}(\omega, t) = \mathbf{A}_{\mathbf{F}}$, a constant matrix, $\mathbf{F}(t)$ is a stationary vector process.

The autocorrelation function of $\mathbf{F}(t)$ is defined by $\mathbf{R}_{\mathbf{FF}}(t, \tau) \equiv E[\mathbf{F}(t)\mathbf{F}^{\mathrm{T}}(t + \tau)]$. Substituting equations (3) and (6) into the previous definition leads to

$$\mathbf{R}_{\mathbf{FF}}(t, \tau) = \int_{-\infty}^{\infty} \mathbf{A}_{\mathbf{F}}^*(\omega, t)\mathbf{\Phi}(\omega)\mathbf{A}_{\mathbf{F}}^{\mathrm{T}}(\omega, t + \tau)e^{\mathrm{j}\omega\tau}\,\mathrm{d}\omega \quad (7)$$

which, for $\tau = 0$, gives

$$E[\mathbf{F}(t)\mathbf{F}^{\mathrm{T}}(t)] \equiv \mathbf{R}_{\mathbf{FF}}(t,0) = \int_{-\infty}^{\infty} \mathbf{A}_{\mathbf{F}}^{*}(\omega,t)\mathbf{\Phi}(\omega)\mathbf{A}_{\mathbf{F}}^{\mathrm{T}}(\omega,t)\,\mathrm{d}\omega. \tag{8}$$

This defines the $m \times m$ evolutionary (time-dependent) power spectral density matrix of $\mathbf{F}(t)$ as

$$\mathbf{\Phi}_{\mathbf{FF}}(\omega,t) = \mathbf{A}_{\mathbf{F}}^{*}(\omega,t)\mathbf{\Phi}(\omega)\mathbf{A}_{\mathbf{F}}^{\mathrm{T}}(\omega,t). \tag{9}$$

A special form of $\mathbf{F}(t)$ is defined as the product of a stationary vector process, $\mathbf{X}_S(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\mathbf{Z}(\omega)$, and a time modulating matrix, $\mathbf{A}_{\mathbf{F}}(t)$, thus reducing equation (3) to

$$\mathbf{F}(t) = \mathbf{A}_{\mathbf{F}}(t)\mathbf{X}_S(t) \tag{10}$$

where $\mathbf{X}_S(t)$ has the power spectral density matrix $\mathbf{\Phi}(\omega)$. Therefore, the evolutionary power spectral density matrix of $\mathbf{F}(t)$ defined in equation (9) takes the separable form

$$\mathbf{\Phi}_{\mathbf{FF}}(\omega,t) = \mathbf{A}_{\mathbf{F}}(t)\mathbf{\Phi}(\omega)\mathbf{A}_{\mathbf{F}}^{\mathrm{T}}(t). \tag{11}$$

In the scalar case, equation (11) reduces to

$$\phi_{\mathrm{FF}}(\omega,t) = a_F^2(t)\phi(\omega) \tag{12}$$

indicating that only the amplitude is changing as a function of time, while the frequency content remains fixed.

An important particular case of separable vector process as defined in equation (10) is obtained when $\mathbf{X}_S(t)$ is a stationary vector white noise process, $\mathbf{W}(t)$, defined by its correlation matrix, $\mathbf{R}_{\mathbf{WW}}(\tau) = \mathbf{R}_0\delta(\tau)$, or power spectral density matrix, $1/(2\pi)\mathbf{R}_0$, where $\mathbf{R}_0$ denotes a constant matrix.

System response

Assuming orthogonal damping and employing the modal superposition method, the relative displacement response vector is expressed as

$$\mathbf{U}(t) = \mathbf{\Phi}\mathbf{Z}(t) \tag{13}$$

where $\mathbf{Z}^{\mathrm{T}}(t) = [Z_1(t), Z_2(t), \ldots, Z_n(t)]$ is the length-n vector of the modal responses and $\mathbf{\Phi} = [\phi_1, \phi_2, \ldots, \phi_n]$ is the $n \times n$ eigenmatrix or matrix of mode shapes. Using the orthogonality properties of the mode shapes with respect to the system mass and stiffness matrices, the equations of motion, equation (1), are transformed into the uncoupled form

$$\ddot{Z}_i(t) + 2\xi_i\omega_i\dot{Z}_i(t) + \omega_i^2 Z_i(t) = \Gamma_i\mathbf{F}(t), \quad i = 1,2,\ldots,n \tag{14}$$

where $\Gamma_i = \phi_i^{\mathrm{T}}\mathbf{P}/(\phi_i^{\mathrm{T}}\mathbf{M}\phi_i)$ is a length-m row vector of the conventional modal participation factors; ω_i and ξ_i denote the undamped modal natural frequencies and damping ratios, respectively. For convenience purposes, it is useful to introduce the length-m normalized modal response vector, $\mathbf{S}_i(t)$, defined by

$$\ddot{\mathbf{S}}_i(t) + 2\xi_i\omega_i\dot{\mathbf{S}}_i(t) + \omega_i^2\mathbf{S}_i(t) = \mathbf{F}(t), \quad i = 1,2,\ldots,n \tag{15}$$

in which $\mathbf{S}_i(t) = [S_{i1}(t), S_{i2}(t), \ldots, S_{im}(t)]^{\mathrm{T}}$ and $S_{ij}(t)$ can be interpreted as the response of a single-degree-of-freedom oscillator of unit mass, undamped natural frequency ω_i and damping ratio ξ_i, to the forcing function $F_j(t)$. It is noticed that the modal response $Z_i(t)$ defined in equation (13) is related to the normalized modal response vector $\mathbf{S}_i(t)$ just defined by

$$Z_i(t) = \Gamma_i\mathbf{S}_i(t). \tag{16}$$

Similarly, in the case of support excitation, the absolute displacement response vector can also be expressed as $\mathbf{X}(t) = \mathbf{\Phi}\mathbf{Y}(t)$ where $\mathbf{Y}^{\mathrm{T}}(t) = [Y_1(t), Y_2(t), \ldots, Y_n(t)]$ and $Y_i(t)$ is the ith modal absolute displacement response. From equation (2), it can be shown that the ith modal absolute displacement response is related to the ith modal relative displacement response by

$$\ddot{Y}_i(t) = -2\xi_i\omega_i\dot{Z}_i(t) - \omega_i^2 Z_i(t) = \Gamma_i\mathbf{S}_i^a(t) \tag{17}$$

where

$$\mathbf{S}_i^a(t) = -2\xi_i\omega_i\dot{\mathbf{S}}_i(t) - \omega_i^2\mathbf{S}_i(t) \tag{18}$$

is the ith normalized modal absolute acceleration response.

For the general class of linear elastic structures, any set of structural response quantities, $\mathbf{Q}(t)$ (e.g. internal forces, stress or strain components, interstory drifts, base shear and overturning moment), can be obtained as a linear combination of the nodal displacement vector relative to the base. Thus

$$\mathbf{Q}(t) = \mathbf{B}\mathbf{U}(t) = \mathbf{B}\mathbf{\Phi}\mathbf{Z}(t) \tag{19}$$

where $\mathbf{Q}(t)$ is a length-l vector of response quantities and $\mathbf{B}$ is a $l \times n$ constant response transfer matrix, which in general is a function of the global or local stiffness and geometric properties of the structure. For example, if the response quantity of interest is the relative displacement of the nth degree of freedom, then $\mathbf{B} = [0, \ldots, 0, 1]$, a length-$n$ row vector.

From equation (15) and assuming zero initial conditions, the Duhamel's integral form of $\mathbf{S}_i(t)$ is given by

$$\mathbf{S}_i(t) = \int_0^t h_i(t-\tau)\mathbf{F}(\tau)\,\mathrm{d}\tau \tag{20}$$

where $h_i(t)$ is the unit-impulse response function for mode i, which for the systems considered is given by

$$h_i(t) = \frac{1}{\omega_{D_i}}\mathrm{e}^{-\xi_i\omega_i t}\sin(\omega_{D_i}t)H(t) \tag{21}$$

where $\omega_{D_i} = \omega_i\sqrt{(1-\xi_i^2)}$ is the damped natural circular frequency of mode i and $H(t)$ is the unit step or Heaviside unit function.

Upon substituting equation (3) into equation (20) and changing the order of integration, the forced vibration response $\mathbf{S}_i(t)$ becomes

$$\mathbf{S}_i(t) = \int_{-\infty}^{\infty}\left[\int_0^t h_i(t-\tau)\mathbf{A_F}(\omega,\tau)\mathrm{e}^{-\mathbf{j}\omega(t-\tau)}\,\mathrm{d}\tau\right]\mathrm{e}^{\mathbf{j}\omega t}\,\mathrm{d}\mathbf{Z}(\omega) = \int_{-\infty}^{\infty}\mathbf{M}_i(\omega,t)\mathrm{e}^{\mathbf{j}\omega t}\,\mathrm{d}\mathbf{Z}(\omega) \quad (22)$$

where

$$\mathbf{M}_i(\omega,t) = \int_0^t h_i(t-\tau)\mathbf{A_F}(\omega,\tau)\mathrm{e}^{-\mathbf{j}\omega(t-\tau)}\,\mathrm{d}\tau \quad (23)$$

is the $m\times k$ matrix of frequency-time modulating functions of $\mathbf{S}_i(t)$ and plays the same role as matrix $\mathbf{A_F}(\omega,t)$ for $\mathbf{F}(t)$, see equation (3). Likewise, the pth time derivative of the ith modal response, if it exists in the mean square sense, can be expressed as

$$\frac{\partial^p}{\partial t^p}\mathbf{S}_i(t) = \mathbf{S}_i^{(p)}(t) = \frac{\partial^p}{\partial t^p}\left[\int_0^t h_i(t-\tau)\mathbf{F}(\tau)\mathrm{d}\tau\right] = \int_{-\infty}^{\infty}\tilde{\mathbf{M}}_i^{(p)}(\omega,t)\mathrm{e}^{\mathbf{j}\omega t}\,\mathrm{d}\mathbf{Z}(\omega) \quad (24)$$

in which

$$\tilde{\mathbf{M}}_i^{(p)}(\omega,t) = \mathrm{e}^{-\mathbf{j}\omega t}\frac{\partial^p}{\partial t^p}[\mathbf{M}_i(\omega,t)\mathrm{e}^{\mathbf{j}\omega t}]. \quad (25)$$

From equation (22), the quantity $\tilde{\mathbf{M}}_i^{(p)}(\omega,t)$ in the above equation can be physically interpreted as the product of $\mathrm{e}^{-\mathbf{j}\omega t}$ and the pth time derivative of the response of the ith modal oscillator subjected to the input force $\mathbf{A_F}(\omega,t)\mathrm{e}^{\mathbf{j}\omega t}$. The $m\times m$ matrix of cross relation functions between derivatives of order p and q of the ith and jth normalized modal responses is given by

$$\mathbf{R}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(t,\tau) = E[\mathbf{S}_i^{(p)}(t)\{\mathbf{S}_j^{(q)}(t+\tau)\}^{\mathrm{T}}] = \int_{-\infty}^{\infty}[\tilde{\mathbf{M}}_i^{(p)}(\omega,t)]^*\mathbf{\Phi}(\omega) \times [\tilde{\mathbf{M}}_j^{(q)}(\omega,t+\tau)]^{\mathrm{T}}\mathrm{e}^{\mathbf{j}\omega\tau}\,\mathrm{d}\omega \quad (26)$$

which, for $\tau = 0$, becomes

$$\mathbf{R}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(t,0) = E[\mathbf{S}_i^{(p)}(t)\{\mathbf{S}_j^{(q)}(t)\}^{\mathrm{T}}] = \int_{-\infty}^{\infty}[\tilde{\mathbf{M}}_i^{(p)}(\omega,t)]^*\mathbf{\Phi}(\omega) \times [\tilde{\mathbf{M}}_j^{(q)}(\omega,t)]^{\mathrm{T}}\mathrm{d}\omega. \quad (27)$$

From equation (26), the $m\times m$ matrix of evolutionary cross-power spectral density functions between derivatives of order p and q of the ith and jth normalized modal responses is

$$\mathbf{\Phi}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(\omega,t) = [\tilde{\mathbf{M}}_i^{(p)}(\omega,t)]^*\mathbf{\Phi}(\omega)[\tilde{\mathbf{M}}_j^{(q)}(\omega,t)]^{\mathrm{T}}. \quad (28)$$

From equation (16), it follows that the cross-correlation and evolutionary power spectral density functions between derivatives of order p and q of the modal responses $Z_i(t)$ and $Z_j(t)$ are

$$R_{Z_i^{(p)}Z_j^{(q)}}(t,\tau) = E\{Z_i^{(p)}(t)[Z_j^{(q)}(t+\tau)]^{\mathrm{T}}\} = \mathbf{\Gamma}_i\mathbf{R}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(t,\tau)\mathbf{\Gamma}_j^{\mathrm{T}} \quad (29)$$

$$\Phi_{Z_i^{(p)}Z_j^{(q)}}(\omega,t) = \mathbf{\Gamma}_i\mathbf{\Phi}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(\omega,t)\mathbf{\Gamma}_j^{\mathrm{T}}. \quad (30)$$

It is worth emphasizing that the summation convention in which a repeated index implies summation over the range of the index is not applied in this paper. Similarly, from equations (13), (29) and (30), it is found that the cross-correlation and evolutionary cross-power spectral density functions between derivatives of order p and q of two nodal response quantities $U_i(t)$ and $U_j(t)$ can be expressed as

$$R_{U_i^{(p)}U_j^{(q)}}(t,\tau) = \sum_{k=1}^{n}\sum_{l=1}^{n}[\phi_{ik}\phi_{jl}\mathbf{\Gamma}_k\mathbf{R}_{\mathbf{S}_k^{(p)}\mathbf{S}_l^{(q)}}(t,\tau)\mathbf{\Gamma}_l^{\mathrm{T}}] \quad (31)$$

$$\Phi_{U_i^{(p)}U_j^{(q)}}(\omega,t) = \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{ik}\phi_{jl}\mathbf{\Gamma}_k\mathbf{\Phi}_{\mathbf{S}_k^{(p)}\mathbf{S}_l^{(q)}}(\omega,t)\mathbf{\Gamma}_l^{\mathrm{T}} \quad (32)$$

in which the coefficient ϕ_{ik} denotes the ith component of the kth mode shape.

Analogous to equations (31) and (32), the cross-correlation and evolutionary cross-power spectral density functions between the nodal absolute acceleration responses $\ddot{X}_i(t)$ and $\ddot{X}_j(t)$ of a ground-excited system are given by the superposition formulas

$$R_{\ddot{X}_i\ddot{X}_j}(t,\tau) = \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{ik}\phi_{jl}\mathbf{\Gamma}_k\mathbf{R}_{\mathbf{S}_k^a\mathbf{S}_l^a}(t,\tau)\mathbf{\Gamma}_l^{\mathrm{T}} \quad (33)$$

$$\phi_{\ddot{X}_i\ddot{X}_j}(\omega,t) = \sum_{k=1}^{n}\sum_{l=1}^{n}\phi_{ik}\phi_{jl}\mathbf{\Gamma}_k\mathbf{\Phi}_{\mathbf{S}_k^a\mathbf{S}_l^a}(\omega,t)\mathbf{\Gamma}_l^{\mathrm{T}} \quad (34)$$

in which, via equation (18), $\mathbf{R}_{\mathbf{S}_k^a\mathbf{S}_l^a}(t,\tau)$ and $\mathbf{\Phi}_{\mathbf{S}_k^a\mathbf{S}_l^a}(\omega,t)$ are determined to be

$$\mathbf{R}_{\mathbf{S}_k^a\mathbf{S}_l^a}(t,\tau) = 4\xi_k\,\xi_l\,\omega_k\,\omega_l\mathbf{R}_{\dot{\mathbf{S}}_k\dot{\mathbf{S}}_l}(t,\tau) + 2\xi_k\,\omega_k\,\omega_l^2\mathbf{R}_{\dot{\mathbf{S}}_k\mathbf{S}_l}(t,\tau) + 2\xi_l\omega_l\omega_k^2\mathbf{R}_{\mathbf{S}_k\dot{\mathbf{S}}_l}(t,\tau) + \omega_k^2\omega_l^2\mathbf{R}_{\mathbf{S}_k\mathbf{S}_l}(t,\tau) \quad (35)$$

$$\mathbf{\Phi}_{\mathbf{S}_k^a\mathbf{S}_l^a}(\omega,t) = 4\xi_k\,\xi_l\,\omega_k\,\omega_l\mathbf{\Phi}_{\dot{\mathbf{S}}_k\dot{\mathbf{S}}_l}(\omega,t) + 2\xi_k\,\omega_k\,\omega_l^2\mathbf{\Phi}_{\dot{\mathbf{S}}_k\mathbf{S}_l}(\omega,t) + 2\xi_l\omega_l\omega_k^2\mathbf{\Phi}_{\mathbf{S}_k\dot{\mathbf{S}}_l}(\omega,t) + \omega_k^2\omega_l^2\mathbf{\Phi}_{\mathbf{S}_k\mathbf{S}_l}(\omega,t). \quad (36)$$

From equation (19), the correlation and evolutionary power spectral density matrices for the generalized structural response vector $\mathbf{Q}(t)$ are expressed as $\mathbf{R_{QQ}}(t,\tau) = \mathbf{BR_{UU}}(t,\tau)\mathbf{B}^{\mathrm{T}}$, $\mathbf{\Phi_{QQ}}(\omega,t) = \mathbf{B\Phi_{UU}}(\omega,t)\mathbf{B}^{\mathrm{T}}$.

Finally, consider a very special case in which a stable system (implying here that $\xi_i > 0$ for $i = 1,\ldots,n$) initially at rest is subjected to a stationary excitation

$\mathbf{F}(t)$ (i.e. $\mathbf{A_F}$ is a constant matrix). In this case, the system's response becomes stationary (i.e. $\mathbf{S}_i(t)$ becomes a stationary vector process) as time tends to infinity. Thus

$$\lim_{t\to\infty} \mathbf{M}_i(\omega, t) = \lim_{t\to\infty} \int_0^t h_i(t-\tau)\mathbf{A_F}e^{-\mathbf{j}\omega(t-\tau)}\,d\tau = \mathbf{A_F}H_i(\omega) = \mathbf{M}_i(\omega) \quad (37)$$

where $H_i(\omega) = \int_0^\infty h_i(\tau)e^{-\mathbf{j}\omega\tau}\,d\tau$ is the Fourier transform of the unit-impulse response function $h_i(t)$ and is called the modal complex frequency response function. From equation (25), it can be shown that

$$\lim_{t\to\infty} \tilde{\mathbf{M}}_i^{(p)}(\omega, t) = (\mathbf{j}\omega)^p\mathbf{A_F}H_i(\omega) = (\mathbf{j}\omega)^p\mathbf{M}_i(\omega) = \tilde{\mathbf{M}}_i^{(p)}(\omega) \quad (38)$$

and the cross-correlation matrix between $\mathbf{S}_i^{(p)}(t)$ and $\mathbf{S}_j^{(q)}(t)$ in equation (27) reduces to

$$\begin{aligned}\mathbf{R}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(t,\tau) &= \mathbf{R}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(\tau) \\ &= \int_{-\infty}^{\infty} [\tilde{\mathbf{M}}_i^{(p)}(\omega)]^*\mathbf{\Phi}(\omega)[\tilde{\mathbf{M}}_j^{(q)}(\omega)]^{\mathrm{T}}e^{\mathbf{j}\omega\tau}\,d\omega \\ &= \int_{-\infty}^{\infty} (-\mathbf{j}\omega)^p(\mathbf{j}\omega)^q\mathbf{A_F}H_i^*(\omega)\mathbf{\Phi}(\omega)H_j(\omega) \times \mathbf{A_F^{\mathrm{T}}}e^{\mathbf{j}\omega\tau}\,d\omega \\ &= \int_{-\infty}^{\infty} \mathbf{\Phi}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(\omega)e^{\mathbf{j}\omega\tau}\,d\omega \end{aligned} \quad (39)$$

where the cross-power spectral density matrix between $\mathbf{S}_i^{(p)}(t)$ and $\mathbf{S}_j^{(q)}(t)$ is

$$\mathbf{\Phi}_{\mathbf{S}_i^{(p)}\mathbf{S}_j^{(q)}}(\omega) = (-\mathbf{j}\omega)^p(\mathbf{j}\omega)^q\mathbf{A_F}H_i^*(\omega)\mathbf{\Phi}(\omega)H_j(\omega)\mathbf{A_F^{\mathrm{T}}}. \quad (40)$$

Equations (39) and (40) are recognized as the well-known Wiener–Khintchine relationship for stationary vector processes.

EXPLICIT CLOSED-FORM SOLUTION FOR CROSS-MODAL CORRELATION AND FREQUENCY–TIME MODULATING FUNCTIONS IN THE CASE OF MODULATED WHITE NOISE EXCITATION

Consider the special case in which the excitation $F(t)$ is a scalar modulated white noise, i.e.

$$F(t) = A_F(t)W(t) \quad (41)$$

where $W(t)$ denotes a zero-mean white noise of constant power spectral density, Φ_{WW}, and autocorrelation function $R_{WW}(\tau) = 2\pi\Phi_{WW}\delta(\tau)$. The present study considers the three-parameter time modulating function

$$A_F(t) = \alpha t^\beta e^{-\lambda t}, \quad \alpha > 0, \quad \beta, \lambda \geqslant 0 \quad (42)$$

proposed by Saragoni and Hart[7] in the context of stochastic earthquake ground motion modeling. It is worth noting that this general time-modulating function includes, as particular cases, the exponentially decaying ($\beta = 0$) and the step ($\beta = \gamma = 0$) modulating functions which have been used in earlier studies on stochastic earthquake response analysis.[6,14] The closed-form solutions presented below apply only when β is a nonnegative integer in the case of the evolutionary power spectral density functions and when 2β is a nonnegative integer in the case of the cross-correlation functions.

From equations (23) and (25), and after some lengthy mathematical manipulations making use of the CRC integral table,[28] it is found that $M_i(\omega, t)$ and $\tilde{M}_i^{(1)}(\omega, t)$ are given by

$$M_i(\omega, t) = \frac{\mathbf{j}\alpha}{2\omega_{D_i}}\left[e^{-\lambda t}\sum_{n=0}^{\beta} E_n(-E_{en1} + E_{en2}) + E_e(E_{eb1} - E_{eb2})\right] \quad (43)$$

$$\begin{aligned}\tilde{M}_i^{(1)}(\omega, t) = \frac{\alpha}{2}\Bigg\{ e^{-\lambda t}\sum_{n=0}^{\beta} E_n\Bigg[\left(1 + \frac{\mathbf{j}\xi_i\omega_i}{\omega_{D_i}}\right)E_{en1} &+ \left(1 - \frac{\mathbf{j}\xi_i\omega_i}{\omega_{D_i}}\right)E_{en2}\Bigg] - E_e\Bigg[\left(1 + \frac{\mathbf{j}\xi_i\omega_i}{\omega_{D_i}}\right)E_{eb1} \\ &+ \left(1 - \frac{\mathbf{j}\xi_i\omega_i}{\omega_{D_i}}\right)E_{eb2}\Bigg]\Bigg\}\end{aligned} \quad (44)$$

where

$$E_n(n, \beta, t) = (-1)^n\beta!t^{(\beta-n)}/(\beta-n)!$$

$$E_{en1}(n, \theta_1, \rho_1) = \exp[-\mathbf{j}(n+1)\theta_1]/\rho_1^{(n+1)}$$

$$E_{en2}(n, \theta_2, \rho_2) = \exp[-\mathbf{j}(n+1)\theta_2]/\rho_2^{(n+1)}$$

$$E_e(\xi_i, \omega_i, \beta, \omega, t) = \exp[-(\mathbf{j}\omega + \xi_i\omega_i)t](-1)^\beta\beta!$$

$$E_{eb1}(\xi_i, \omega_i, \beta, \theta_1, \rho_1, t) = \exp\{\mathbf{j}[\omega_{D_i}t - (\beta+1)\theta_1]\}/\rho_1^{(\beta+1)}$$

$$E_{eb2}(\xi_i, \omega_i, \beta, \theta_2, \rho_2, t) = \exp\{-\mathbf{j}[\omega_{D_i}t + (\beta+1)\theta_2]\}/\rho_2^{(\beta+1)}$$

$$\rho_1 = \sqrt{(\xi_i\omega_i - \lambda)^2 + (\omega - \omega_{D_i})^2}$$

$$\rho_1\cos\theta_1 = \xi_i\omega_i - \lambda \quad \rho_1\sin\theta_1 = \omega - \omega_{D_i}$$

$$\rho_2 = \sqrt{(\xi_i\omega_i - \lambda)^2 + (\omega + \omega_{D_i})^2}$$

$$\rho_2\cos\theta_2 = \xi_i\omega_i - \lambda \quad \rho_2\sin\theta_2 = \omega + \omega_{D_i}.$$

It should be noted here that both equations (43) and (44) can also be used to compute the evolutionary power spectral density matrix of the response vector for any gamma-modulated stationary input process using equations (28), (30), (32), (34) and (36).

The analytical forms of the modal cross-correlation functions can be derived directly from the definition

$$R_{S_i^{(p)}S_j^{(q)}}(t,\tau) = E[S_i^{(p)}(t)S_j^{(q)}(t+\tau)] \tag{45}$$

and by using equation (20). Only the case $\tau \geqslant 0$ is considered below, since for $\tau < 0$ the following relation can be used

$$R_{S_i^{(p)}S_j^{(q)}}(t,\tau) = R_{S_j^{(q)}S_i^{(p)}}(t+\tau,-\tau), \quad (t+\tau) \geqslant 0. \tag{46}$$

After extensive mathematical derivations, the following explicit closed-form expressions are obtained:

$$R_{S_iS_j}(t,\tau) = \frac{\pi\Phi_{WW}\alpha^2}{\omega_{D_i}\omega_{D_j}} \times \left\{ G_{e1}\sum_{n=0}^{2\beta} G_n(G_{cn1}-G_{cn2}) - G_{e2}(G_{c\beta1}-G_{c\beta2})\right\} \tag{47}$$

$$R_{\dot{S}_iS_j}(t,\tau) = \frac{\pi\Phi_{WW}\alpha^2}{\omega_{D_j}} \times \left\{ G_{e1}\sum_{n=0}^{2\beta} G_n\left[G_{sn1}+G_{sn2}-\frac{\xi_i\omega_i}{\omega_{D_i}}(G_{cn1}-G_{cn2})\right] - G_{e2}\left[-G_{s\beta1}+G_{s\beta2}-\frac{\xi_i\omega_i}{\omega_{D_i}}(G_{c\beta1}-G_{c\beta2})\right]\right\} \tag{48}$$

$$R_{S_i\dot{S}_j}(t,\tau) = \frac{\pi\Phi_{WW}\alpha^2}{\omega_{D_i}} \times \left\{ G_{e1}\sum_{n=0}^{2\beta} G_n\left[-G_{sn1}+G_{sn2}-\frac{\xi_j\omega_j}{\omega_{D_j}}(G_{cn1}-G_{cn2})\right] - G_{e2}\left[G_{s\beta1}+G_{s\beta2}-\frac{\xi_j\omega_j}{\omega_{D_j}}(G_{c\beta1}-G_{c\beta2})\right]\right\} \tag{49}$$

$$\begin{aligned} R_{\dot{S}_i\dot{S}_j}(t,\tau) &= \pi\Phi_{WW}\alpha^2 \\ &\times \left\{ G_{e1}\sum_{n=0}^{2\beta} G_n\left[\left(\frac{\xi_i\omega_i}{\omega_{D_i}}-\frac{\xi_j\omega_j}{\omega_{D_j}}\right)G_{sn1} - \left(\frac{\xi_i\omega_i}{\omega_{D_i}}+\frac{\xi_j\omega_j}{\omega_{D_j}}\right)G_{sn2} + \left(1+\frac{\xi_i\omega_i\xi_j\omega_j}{\omega_{D_i}\omega_{D_j}}\right)G_{cn1} \right.\right. \\ &\left. + \left(1-\frac{\xi_i\omega_i\xi_j\omega_j}{\omega_{D_i}\omega_{D_j}}\right)G_{cn2}\right] - G_{e2}\left[\left(1+\frac{\xi_i\omega_i\xi_j\omega_j}{\omega_{D_i}\omega_{D_j}}\right)G_{c\beta1} + \left(1-\frac{\xi_i\omega_i\xi_j\omega_j}{\omega_{D_i}\omega_{D_j}}\right)G_{c\beta2} + \left(-\frac{\xi_i\omega_i}{\omega_{D_i}}+\frac{\xi_j\omega_j}{\omega_{D_j}}\right)G_{s\beta1} \right. \\ &\left.\left. + \left(\frac{\xi_i\omega_i}{\omega_{D_i}}+\frac{\xi_j\omega_j}{\omega_{D_j}}\right)G_{s\beta2}\right]\right\} \end{aligned} \tag{50}$$

where

$$G_{e1}(\lambda,\xi_j,\omega_j,t,\tau) = e^{(-2\lambda t-\xi_j\omega_j\tau)}$$

$$G_{e2}(\xi_i,\omega_i,\xi_j,\omega_j,t,\tau) = (-1)^{(2\beta)}(2\beta)!e^{[-\xi_i\omega_i t-\xi_j\omega_j(t+\tau)]}$$

$$G_n(n,\beta,t) = (-1)^n(2\beta)!t^{(2\beta-n)}/(2\beta-n)!$$

$$G_{cn1}(n,\rho_1,\theta_1,\omega_{D_j},\tau) = \cos[\omega_{D_j}\tau-(n+1)\theta_1]/\rho_1^{(n+1)}$$

$$G_{cn2}(n,\rho_2,\theta_2,\omega_{D_j},\tau) = \cos[\omega_{D_j}\tau+(n+1)\theta_2]/\rho_2^{(n+1)}$$

$$G_{sn1}(n,\rho_1,\theta_1,\omega_{D_j},\tau) = \sin[\omega_{D_j}\tau-(n+1)\theta_1]/\rho_1^{(n+1)}$$

$$G_{sn2}(n,\rho_2,\theta_2,\omega_{D_j},\tau) = \sin[\omega_{D_j}\tau+(n+1)\theta_2]/\rho_2^{(n+1)}$$

$$G_{c\beta1}(\beta,\rho_1,\theta_1,\omega_{D_i},\omega_{D_j},t,\tau) = \cos[(\omega_{D_i}-\omega_{D_j})t-\omega_{D_j}\tau + (2\beta+1)\theta_1]/\rho_1^{(2\beta+1)}$$

$$G_{c\beta2}(\beta,\rho_2,\theta_2,\omega_{D_i},\omega_{D_j},t,\tau) = \cos[(\omega_{D_i}+\omega_{D_j})t+\omega_{D_j}\tau + (2\beta+1)\theta_2]/\rho_2^{(2\beta+1)}$$

$$G_{s\beta1}(\beta,\rho_1,\theta_1,\omega_{D_i},\omega_{D_j},t,\tau) = \sin[(\omega_{D_i}-\omega_{D_j})t-\omega_{D_j}\tau + (2\beta+1)\theta_1]/\rho_1^{(2\beta+1)}$$

$$G_{s\beta2}(\beta,\rho_2,\theta_2,\omega_{D_i},\omega_{D_j},t,\tau) = \sin[(\omega_{D_i}+\omega_{D_j})t + \omega_{D_j}\tau+(2\beta+1)\theta_2]/\rho_2^{(2\beta+1)}$$

$$\rho_1 = \sqrt{(\xi_i\omega_i+\xi_j\omega_j-2\lambda)^2+(\omega_{D_i}-\omega_{D_j})^2}$$

$$\rho_1\cos\theta_1 = \xi_i\omega_i+\xi_j\omega_j-2\lambda \quad \rho_1\sin\theta_1 = \omega_{D_i}-\omega_{D_j}$$

$$\rho_2 = \sqrt{(\xi_i\omega_i+\xi_j\omega_j-2\lambda)^2+(\omega_{D_i}+\omega_{D_j})^2}$$

$$\rho_2\cos\theta_2 = \xi_i\omega_i+\xi_j\omega_j-2\lambda \quad \rho_2\sin\theta_2 = \omega_{D_i}+\omega_{D_j}.$$

Next, it is shown that the above nonstationary solution reduces to the well-known solution for the stationary modal response to white noise excitation. For this purpose, the case ($\alpha = 1$, $\beta = 0$, $\lambda = 0$, and $t \to \infty$) is considered. By substituting these parameter values in equations (43) and (44) and after some algebraic manipulations, it is found that

$$M_i(\omega) = \frac{\mathrm{j}}{2\omega_{D_i}}\left[-\frac{e^{-\mathrm{j}\theta_1}}{\rho_1}+\frac{e^{-\mathrm{j}\theta_2}}{\rho_2}\right] = H_i(\omega) \tag{51}$$

$$\tilde{M}_i^{(1)}(\omega) = \frac{1}{2}\left[\left(1+\frac{\mathrm{j}\xi_i\omega_i}{\omega_{D_i}}\right)\frac{e^{-\mathrm{j}\theta_1}}{\rho_1}+\left(1-\frac{\mathrm{j}\xi_i\omega_i}{\omega_{D_i}}\right)\frac{e^{-\mathrm{j}\theta_2}}{\rho_2}\right] = \mathrm{j}\omega H_i(\omega) \tag{52}$$

in which

$$\rho_1 = \sqrt{(\xi_i\omega_i)^2+(\omega-\omega_{D_i})^2} \qquad \rho_1\cos\theta_1 = \xi_i\omega_i$$

$$\rho_1\sin\theta_1 = \omega-\omega_{D_i}$$

$$\rho_2 = \sqrt{(\xi_i\omega_i)^2+(\omega+\omega_{D_i})^2} \qquad \rho_2\cos\theta_2 = \xi_i\omega_i$$

$$\rho_2\sin\theta_2 = \omega+\omega_{D_i}.$$

Then equation (28) becomes

$$\lim_{t\to\infty} \Phi_{S_iS_i}(\omega,t) = H_i^*(\omega)\Phi_{WW}H_i(\omega) = |H_i(\omega)|^2\Phi_{WW}. \tag{53}$$

Similarly, in the time domain, equations (47)–(50) reduce to

$$R_{S_iS_i}(\tau) = \frac{\pi\Phi_{WW}\alpha^2}{\omega_{D_i}\omega_{D_i}} G_{e1}G_n(G_{cn1} - G_{cn2})$$
$$= \frac{\pi\Phi_{WW}\mathrm{e}^{-\xi_i\omega_i\tau}}{2\xi_i\omega_i^3}\left[\cos(\omega_{D_i}\tau) + \frac{\xi_i}{\sqrt{1-\xi_i^2}}\sin(\omega_{D_i}\tau)\right] \tag{54}$$

$$R_{\dot{S}_iS_i}(\tau)$$
$$= \frac{\pi\Phi_{WW}\alpha^2}{\omega_{D_i}} G_{e1}G_n\left[G_{sn1} + G_{sn2} - \frac{\xi_i\omega_i}{\omega_{D_i}}(G_{cn1} - G_{cn2})\right]$$
$$= \frac{\pi\Phi_{WW}\mathrm{e}^{-\xi_i\omega_i\tau}\sin(\omega_{D_i}\tau)}{2\xi_i\omega_i^2\sqrt{1-\xi_i^2}} \tag{55}$$

$$R_{S_i\dot{S}_i}(\tau)$$
$$= \frac{\pi\Phi_{WW}\alpha^2}{\omega_{D_i}} G_{e1}G_n\left[-G_{sn1} + G_{sn2} - \frac{\xi_i\omega_i}{\omega_{D_i}}(G_{cn1} - G_{cn2})\right]$$
$$= \frac{-\pi\Phi_{WW}\mathrm{e}^{-\xi_i\omega_i\tau}\sin(\omega_{D_i}\tau)}{2\xi_i\omega_i^2\sqrt{1-\xi_i^2}} \tag{56}$$

$$R_{\dot{S}_i\dot{S}_i}(\tau) = \pi\Phi_{WW}\alpha^2 G_{e1}G_n\left[-2\left(\frac{\xi_i\omega_i}{\omega_{D_i}}\right)G_{sn2}\right.$$
$$\left. + \left(1 + \left(\frac{\xi_i\omega_i}{\omega_{D_i}}\right)^2\right)G_{cn1} + \left(1 - \left(\frac{\xi_i\omega_i}{\omega_{D_i}}\right)^2\right)G_{cn2}\right]$$
$$= \frac{\pi\Phi_{WW}\mathrm{e}^{-\xi_i\omega_i\tau}}{2\xi_i\omega_i}\left[\cos(\omega_{D_i}\tau) - \frac{\xi_i}{\sqrt{1-\xi_i^2}}\sin(\omega_{D_i}\tau)\right] \tag{57}$$

in which

$$G_{e1} = \mathrm{e}^{-\xi_i\omega_i\tau} \quad G_n = 1 \quad G_{cn1} = \cos(\omega_{D_i}\tau)/(2\xi_i\omega_i)$$

$$G_{cn2} = \left[\xi_i\cos(\omega_{D_i}\tau) - \sqrt{1-\xi_i^2}\sin(\omega_{D_i}\tau)\right]/(2\omega_i)$$

$$G_{sn1} = \sin(\omega_{D_i}\tau)/(2\xi_i\omega_i)$$

$$G_{sn2} = \left[\xi_i\sin(\omega_{D_i}\tau) + \sqrt{1-\xi_i^2}\cos(\omega_{D_i}\tau)\right]/(2\omega_i)$$

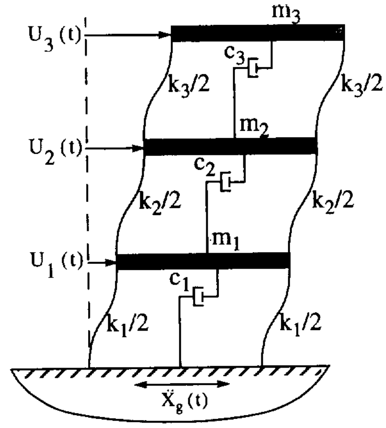

Fig. 1. A three-story shear building model.

Equations (53)–(57) are recognized as the well-known stationary results for a SDOF oscillator excited by white noise.

APPLICATION TO A SIMPLE MDOF STRUCTURE

Consider the horizontal translational vibration of a three-story building structure subjected to earthquake ground motion as shown in Fig. 1. The earthquake ground acceleration is modeled as a modulated white noise with the time modulating function given in equation (42). The different members of this general class of modulating functions selected for this study are portrayed in Fig. 2 and their parameters are given in Table 1. They are normalized according to the L_2-norm such that $\int_0^{60} A_F^2(t)\,\mathrm{d}t = 1$. When β and λ equal zero, the modulating function reduces to the step function represented as Case I in Fig. 2. If only β equals zero,

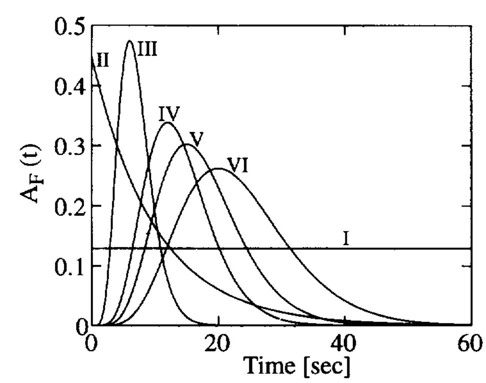

Fig. 2. Modulating functions.

Table 1. Parameters of the modulating functions

Case	I	II	III	IV	V	VI
α	$1/\sqrt{60}$	$\sqrt{0{\cdot}2}$	0·0041	$4{\cdot}569 \times 10^{-5}$	$1{\cdot}071 \times 10^{-5}$	$1{\cdot}651 \times 10^{-6}$
β	0	0	6	6	6	6
γ	0	0·1	1	0·5	0·4	0·3

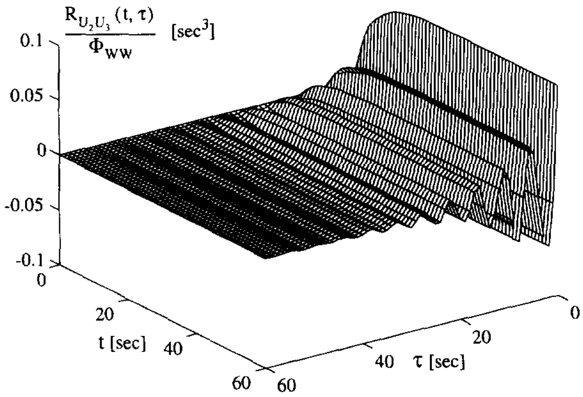

Fig. 3. Cross-correlation function between relative displacement responses $U_2(t)$ and $U_3(t)$ for case-I modulated white noise.

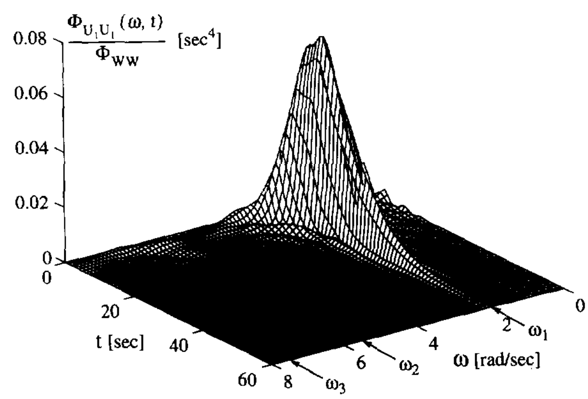

Fig. 5. Evolutionary auto-spectrum function of relative displacement response $U_1(t)$ for case-II modulated white noise.

the modulating function is of the exponential type or case II in Fig. 2.

The building structure under consideration is modeled as a lumped-mass shear building, see Fig. 1. It is assumed that the story stiffnesses are all equal ($k_1 = k_2 = k_3 = k$) and that the lumped floor masses satisfy $m_1 = m_2 = 2m_3 = m$. A ratio of $k/m = 14{\cdot}928$ is selected and leads to three undamped natural circular frequencies of 2, 5·464 and 7·464 [rad/s]. Rayleigh damping is assumed with the mass and stiffness proportionality coefficients taken as $c_m = 0{\cdot}15$ and $c_k = 0{\cdot}01$, respectively, which results in the modal damping ratios $\xi_1 = 4{\cdot}75\%$, $\xi_2 = 4{\cdot}10\%$, and $\xi_3 = 4{\cdot}74\%$.

Figure 3 displays the cross-correlation function between the relative floor displacements $U_2(t)$ and $U_3(t)$ for the case-I modulating function (step function). It is observed that, in agreement with equations (54)–(57), the response gradually becomes stationary and conjunctively the cross-correlation function becomes a function of the time lag τ only. Figure 4 represents the plot of the auto-correlation function of the absolute acceleration $\ddot{X}_3(t)$ or roof absolute acceleration for the case-V modulating function. As expected, the nonstationary response to this modulated white noise does not reach stationarity. The auto-correlation function behaves like the corresponding modulating function along the time axis t and decays with an oscillatory behavior along the time lag axis τ. At zero time lag, the auto-correlation function reduces to the mean square response. A close examination of the results indicates that the maximum value of the mean square response is slightly delayed (3·5 s) with respect to the time at which the modulating function reaches its maximum. This time lag between the nonstationary input and response was first noticed by Spanos[13] and Solomos and Spanos[29] for SDOF oscillators. Using stochastic averaging, they proved that the mean-square response could be approximated by a simple convolution integral of the modulation function. Later, by extending their approximate analytical results to include time-varying statistics of the response PSD and considering multi-mode structures, Igusa[30,31] and Xu and Igusa[32] observed the same time lag phenomenon. Figure 5 shows the evolutionary auto-spectrum of the first floor relative displacement, $U_1(t)$, when the building model is subjected to case-II modulated white noise (exponentially decaying type). Along the circular frequency axis ω, the evolutionary spectrum exhibits local maxima at the natural circular frequencies of the system $(\omega_1, \omega_2, \omega_3)$. The auto-spectrum of the first floor

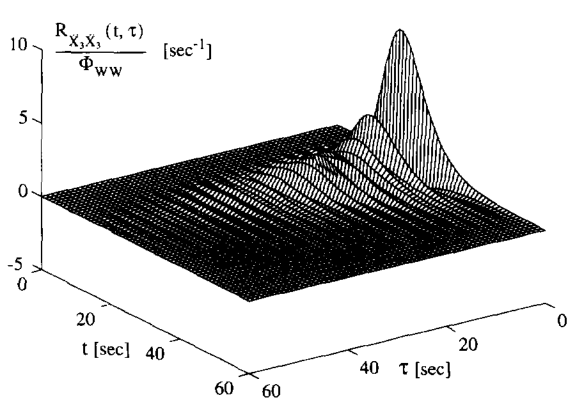

Fig. 4. Auto-correlation function of absolute acceleration response $\ddot{X}_3(t)$ for case-V modulated white noise.

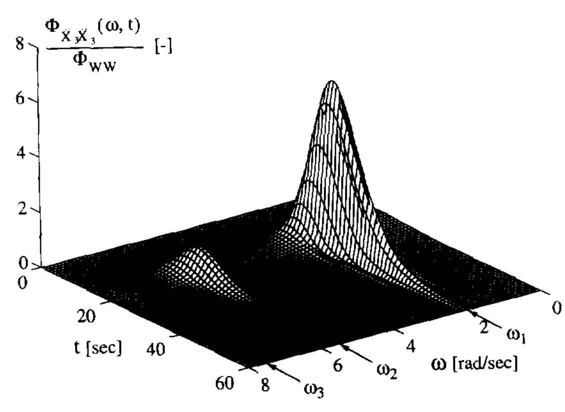

Fig. 6. Evolutionary auto-spectrum function of absolute acceleration response $\ddot{X}_3(t)$ for case-V modulated white noise.

relative displacement reaches its maximum at the first natural frequency of the structure due to the predominant contribution of the first mode in this response quantity. The effective participation factor of the highest (third) mode of the system is so small compared to that of the first two modes that the local maximum of the auto-spectrum at the third natural frequency is not even observable. It is worth noting that since the ground excitation is uniformly modulated (i.e. fixed frequency content) and the system is time invariant, the locations of the spectral peaks along the frequency axis do not change with time. Along the time axis, the shape of the auto-spectrum resembles that of the excitation time modulating function, except at the beginning of the excitation during the response build-up. Figure 6 presents the evolutionary auto-spectrum of the roof absolute acceleration, $\ddot{X}_3(t)$, corresponding to case-V modulated white noise excitation (gamma type). Note that similar to equation (8) for the random excitation process, the following equation holds for the response process $\ddot{X}_3(t)$:

$$R_{\ddot{X}_3\ddot{X}_3}(t,0) = \int_{-\infty}^{\infty} |\Phi_{\ddot{X}_3\ddot{X}_3}(\omega,t)|^2 \mathrm{d}\omega. \tag{58}$$

The above equation expresses the relationship existing between the mean square of the response process $\ddot{X}_3(t)$ and its evolutionary auto-spectrum, namely that the auto-spectrum represents the time-frequency distribution of the mean square response or response energy. By comparing Fig. 6 with Figs 4 and 17, one can observe the similarity between the shape of the mean square of the response $\ddot{X}_3(t)$ and the shape of the corresponding evolutionary auto-spectrum along the time axis at the natural frequencies of the system around which the energy of the response is concentrated.

As an illustration of the more general case of uniformly modulated stationary input process, Figs 7 and 8 display the evolutionary auto-spectrum functions of the relative displacement response $U_1(t)$ and absolute acceleration response $\ddot{X}_3(t)$ for the case-II and case-V modulating functions, respectively. The stationary process considered has an exponentially decaying harmonic correlation function, $R(\tau) = e^{-\nu|\tau|}\cos(\omega_0\tau)$, in which $\omega_0 = 5$ [rad/s] and $\nu = 0{\cdot}5$ [s^{-1}] are selected. The corresponding power spectral density function is

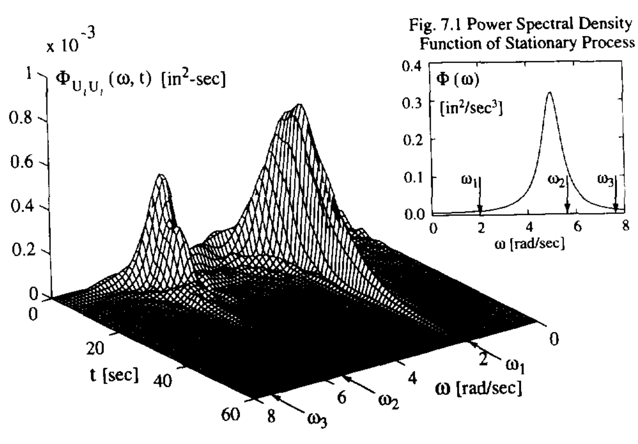

Fig. 7. Evolutionary auto-spectrum function of relative displacement response $U_1(t)$ for case-II modulated stationary process with an exponentially decaying harmonic correlation function.

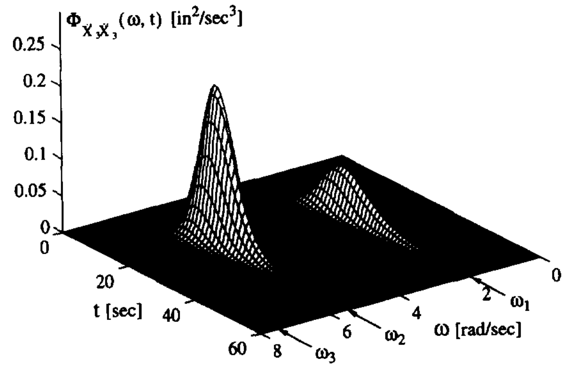

Fig. 8. Evolutionary auto-spectrum function of absolute acceleration response $\ddot{X}_3(t)$ for case-V modulated stationary process with an exponentially decaying harmonic correlation function.

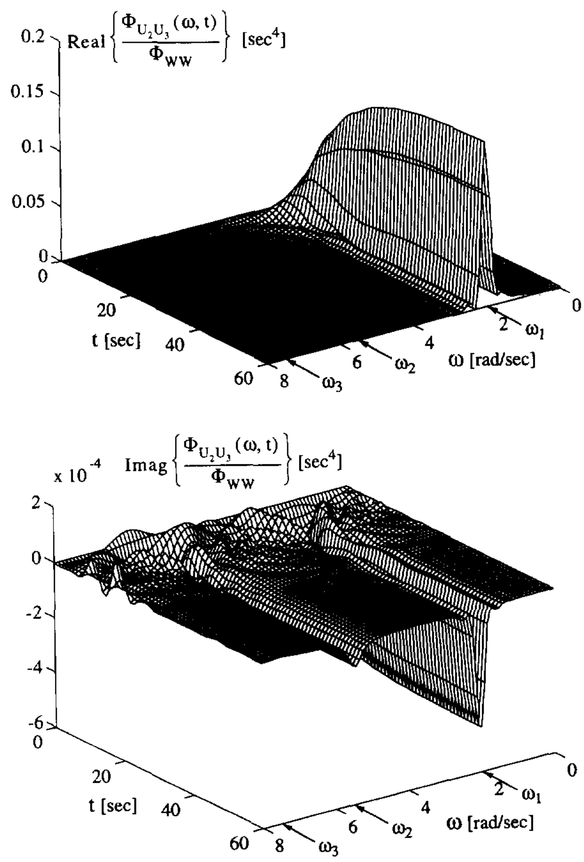

Fig. 9. Evolutionary cross-spectrum function between relative displacement responses $U_2(t)$ and $U_3(t)$ for case-I modulated white noise.

given by

$$\Phi(\omega) = \frac{\nu}{2\pi}\left[\frac{1}{\nu^2 + (\omega + \omega_0)^2} + \frac{1}{\nu^2 + (\omega - \omega_0)^2}\right]$$

and is shown in Fig. 7.1. In comparison with Fig. 5, Fig. 7 indicates that the second mode contribution to the mean square response $U_1(t)$ is significantly increased due to the concentration of the input energy around the frequency $\omega_0 = 5$ rad/s, which is closer to the second than to the first circular natural frequency of the building. The same result is observed in Fig. 8 in which the second mode dominates the mean square of the absolute acceleration response $\ddot{X}_3(t)$. This phenomenon shows the important effects on structural response of the frequency content of the ground motion in relation to the structural natural frequencies.

The evolutionary cross-spectrum between the relative displacement responses $U_2(t)$ and $U_3(t)$ to the case-I modulated white noise is plotted in Fig. 9. As in the case of the cross-correlation function shown in Fig. 3, both the real and imaginary parts of the cross-spectrum become progressively functions of the frequency ω only, while the response gradually reaches stationarity. In the small time range, notice that the imaginary part of the cross-spectrum is much more irregular than the real part thereof. However, in this particular case, the amplitude of the complex-valued cross-spectrum is very close to the real part which is several orders of magnitude larger than the imaginary part.

The plot of the evolutionary cross-spectrum between the absolute acceleration responses $\ddot{X}_1(t)$ and $\ddot{X}_3(t)$ to the white noise ground acceleration modulated with the gamma function (case-V) is presented in Fig. 10. Notice that the imaginary part of the cross-spectrum is much smoother than in the case displayed in Fig. 9. This is due to the absence of discontinuity (at $t = 0$) in the modulating function of the excitation, contrary to cases I (step modulating function) and II (exponentially decaying modulating function) as shown in Fig. 2. In this case again, the real part of the cross-spectrum dominates the imaginary part in amplitude. However, from Fig. 10, it appears that, contrary to the case of the step modulating function (Fig. 9), the contribution of the imaginary part of the cross-spectrum to the amplitude of the cross-spectrum is not negligible in the case of the gamma modulating function (case-V).

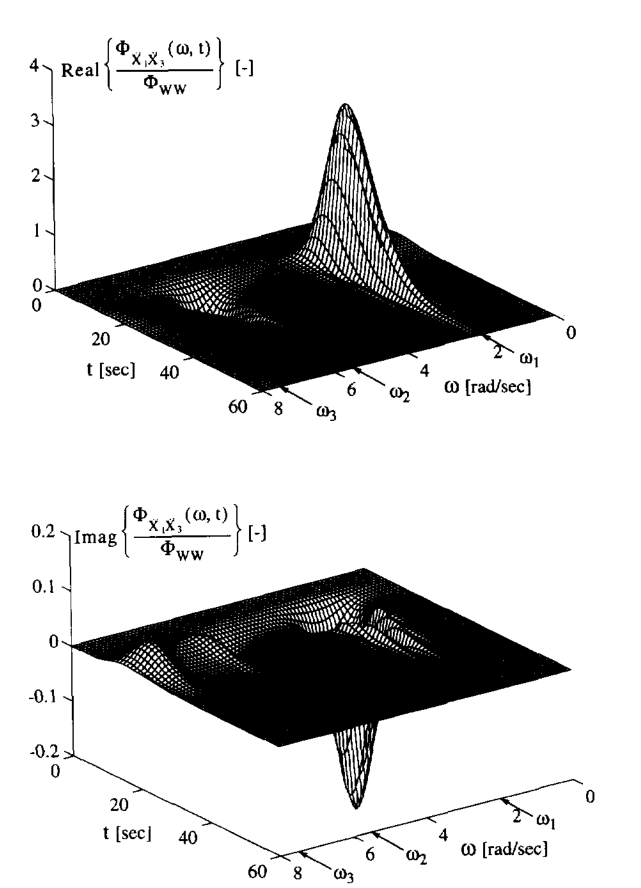

Fig. 10. Evolutionary cross-spectrum function between absolute acceleration responses $\ddot{X}_1(t)$ and $\ddot{X}_3(t)$ for case-V modulated white noise.

EFFECTS OF STATISTICAL CORRELATION BETWEEN MODAL RESPONSES

It was demonstrated by Der Kiureghian[33] that, in computing the statistics of the stationary response of MDOF structures subjected to wideband excitations, cross-correlation terms between modal responses are only significant in the case of systems with closely spaced modes. Otherwise, these cross-modal terms can be neglected and in particular for lightly damped systems. In this paper, the effects of the cross terms between modal responses is investigated in the case of the nonstationary response to modulated white noise with the modulating functions defined earlier, see equation (42).

The importance of modal correlation is quantified through the following cross-modal correlation coefficients:

$$\rho_{0,ij}(t) = \frac{R_{S_iS_j}(t,0)}{\sqrt{R_{S_iS_i}(t,0)R_{S_jS_j}(t,0)}} \quad (59)$$

$$\rho_{2,ij}(t) = \frac{R_{\dot{S}_i\dot{S}_j}(t,0)}{\sqrt{R_{\dot{S}_i\dot{S}_i}(t,0)R_{\dot{S}_j\dot{S}_j}(t,0)}} \quad (60)$$

$$\rho_{a,ij}(t) = \frac{R_{S_i^aS_j^a}(t,0)}{\sqrt{R_{S_i^aS_i^a}(t,0)R_{S_j^aS_j^a}(t,0)}} \quad (61)$$

$$\rho_{1,ij}(t) = \frac{R_{\dot{S}_iS_j}(t,0)}{\sqrt{R_{\dot{S}_i\dot{S}_i}(t,0)R_{S_jS_j}(t,0)}} \quad (62)$$

The coefficients $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$, and $\rho_{a,ij}(t)$ can be physically interpreted as the correlation coefficients between the normalized modal displacement responses $S_i(t)$ and $S_j(t)$, between the normalized modal velocity

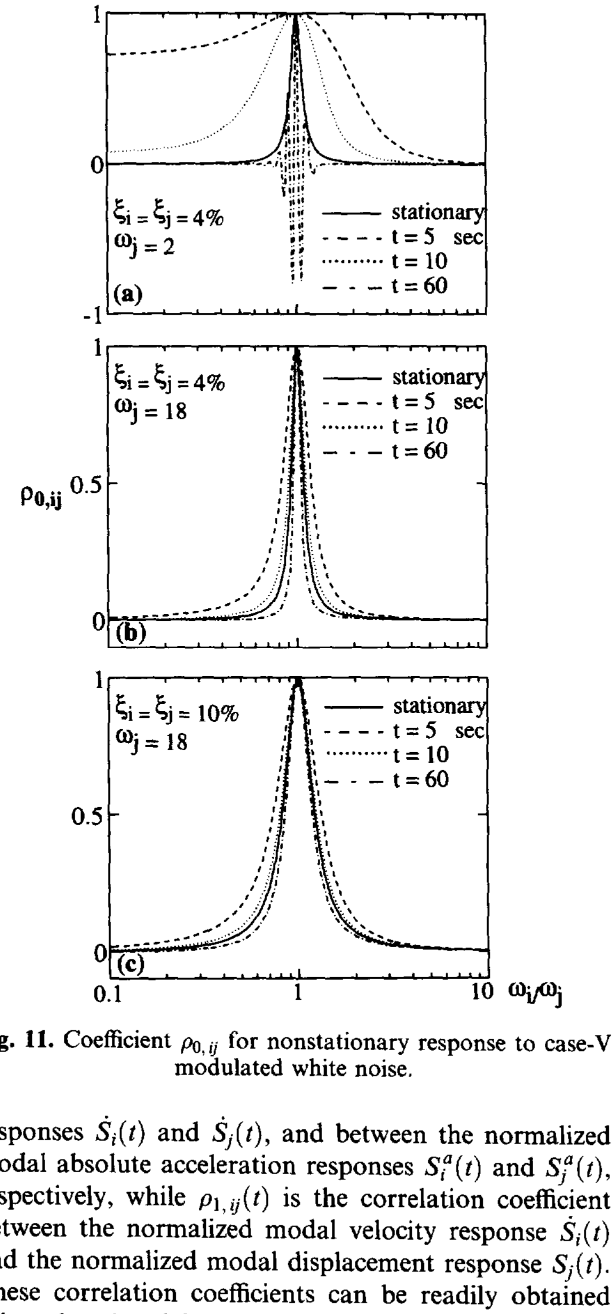

Fig. 11. Coefficient $\rho_{0,ij}$ for nonstationary response to case-V modulated white noise.

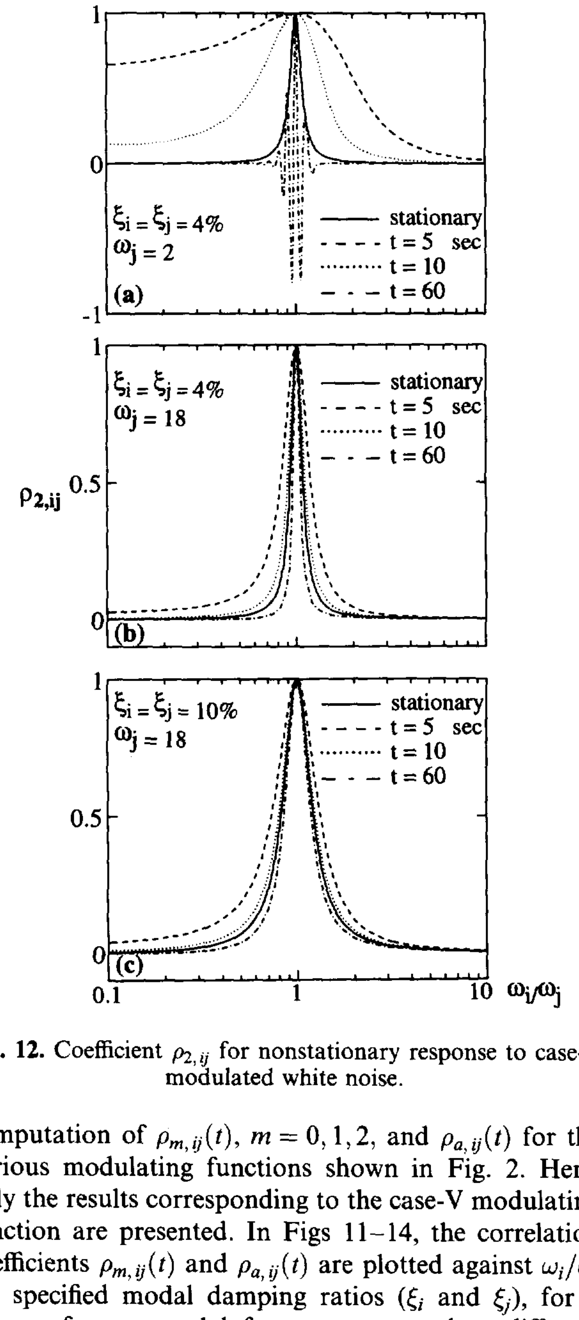

Fig. 12. Coefficient $\rho_{2,ij}$ for nonstationary response to case-V modulated white noise.

responses $\dot{S}_i(t)$ and $\dot{S}_j(t)$, and between the normalized modal absolute acceleration responses $S_i^a(t)$ and $S_j^a(t)$, respectively, while $\rho_{1,ij}(t)$ is the correlation coefficient between the normalized modal velocity response $\dot{S}_i(t)$ and the normalized modal displacement response $S_j(t)$. These correlation coefficients can be readily obtained using the closed-form solutions derived in equations (47)–(50) for the cross-modal correlation functions. As expected, when $i = j$, the correlation coefficients $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$ and $\rho_{a,ij}(t)$ take the unit value as shown in Figs 11–13. Notice that the correlation coefficient $\rho_{1,ij}(t)$ plotted in Fig. 14 is not zero as in stationary random vibrations, and that its absolute value is maximum both below and above the value $\omega_i/\omega_j = 1$. Plots of these modal correlation coefficients as a function of the modal frequency ratio ω_i/ω_j for specified modal damping ratios ξ_i and ξ_j (see Figs 11–14) indicate when the effects of statistical modal correlation are important and when they are negligible in computing mean square response quantities as well as cross-correlation between any two response quantities at the same time.

A comprehensive parametric study was conducted via computation of $\rho_{m,ij}(t)$, $m = 0, 1, 2$, and $\rho_{a,ij}(t)$ for the various modulating functions shown in Fig. 2. Here, only the results corresponding to the case-V modulating function are presented. In Figs 11–14, the correlation coefficients $\rho_{m,ij}(t)$ and $\rho_{a,ij}(t)$ are plotted against ω_i/ω_j for specified modal damping ratios (ξ_i and ξ_j), for a given reference modal frequency ω_j, and at different times. In general, the cross-terms $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$ and $\rho_{a,ij}(t)$ decay rapidly as the two modal frequencies ω_i and ω_j depart from one another and after a sufficiently long time. Otherwise, for closely-spaced modal frequencies or at small time, the cross-modal contributions to the system response statistics cannot be neglected. The width of the region along the ω_i/ω_j axis over which the cross-modal terms are not negligible is usually largest at the beginning of the response and decreases over time. The width of this region depends on the reference modal frequency ω_j; it decreases for increasing ω_j. It is also found that during the early stage of the response to case-I (step function) and case-II (exponential-type) modulating functions and for a relatively small reference frequency ω_j (e.g. $\omega_j = 2 \div 4$ rad/s), the modal

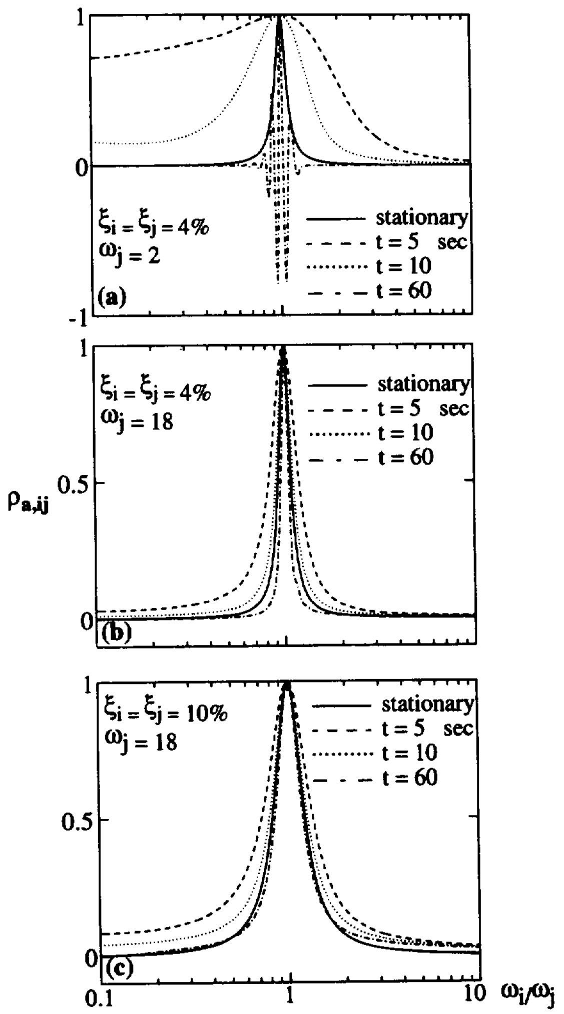

Fig. 13. Coefficient $\rho_{a,ij}$ for nonstationary response to case-V modulated white noise.

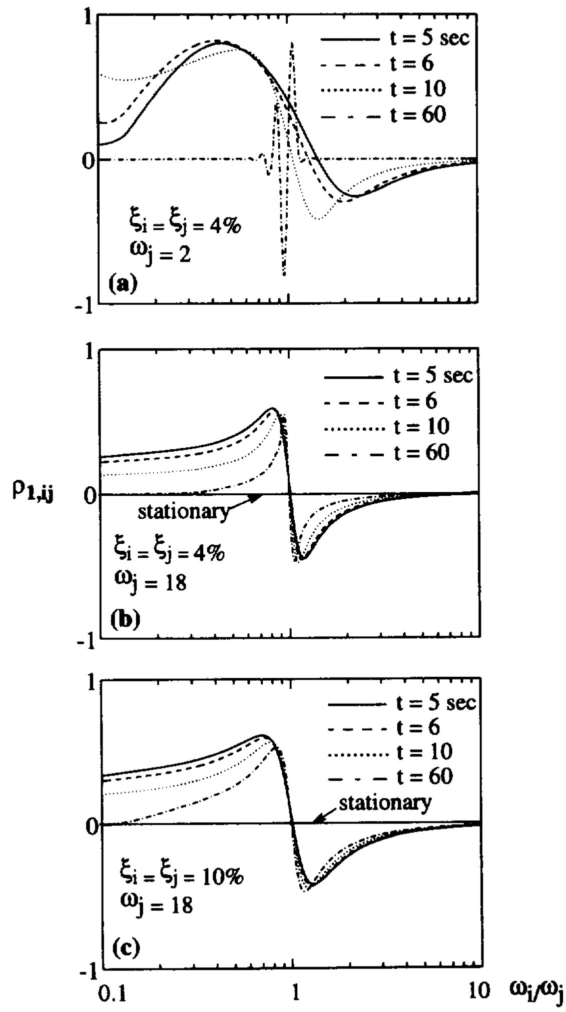

Fig. 14. Coefficient $\rho_{1,ij}$ for nonstationary response to case-V modulated white noise.

correlation coefficients exhibit a decaying oscillatory behavior. This oscillatory behavior is probably due to the discontinuity of the case-I and case-II modulating functions at time zero. Notice also that the cross-modal correlation coefficients $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$ and $\rho_{a,ij}(t)$ are only rarely negative such as in the cases just mentioned and in the cases shown in Figs 11(a), 12(a) and 13(a) at the late stage of the response. It is worth noting that $\rho_{1,ij}(t)$ does not decay as fast as $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$ and $\rho_{a,ij}(t)$ away from $\omega_i/\omega_j = 1$. As in the case of stationary response to wideband stationary input,[33] $\rho_{m,ij}(t)$, $m = 0, 2$, and $\rho_{a,ij}(t)$ decay faster for smaller damping values as the ratio ω_i/ω_j departs from one. Notice that the cross-modal correlation coefficients $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$ and $\rho_{a,ij}(t)$ corresponding to the stationary case for white noise excitation (i.e. case-I modulating function after a sufficiently long time) have been included in Figs 11–13 (solid lines) for comparison with the non-stationary case considered here. It is observed that in the early part of the response process, the "nonstationary" cross-modal correlation coefficients $\rho_{0,ij}(t)$, $\rho_{2,ij}(t)$ and $\rho_{a,ij}(t)$ are larger than those for the stationary case, while the reverse is true during the later part of the response.

Figure 15 represents the mean square relative displacement responses for the three stories, both exact (solid lines) and approximate (dotted lines) by neglecting the cross-modal terms (i.e. assuming $\rho_{0,ij}(t) = 0$ for

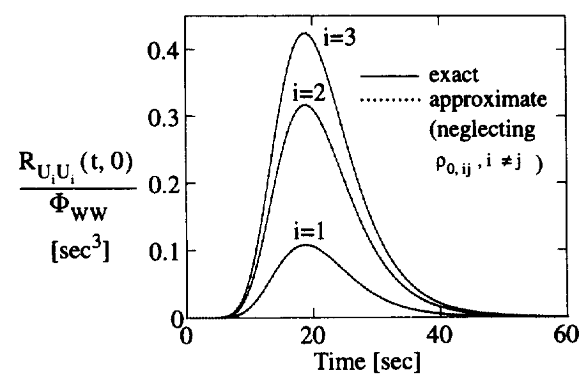

Fig. 15. Mean square relative displacement responses $U_i(t)$ to case-V modulated white noise.

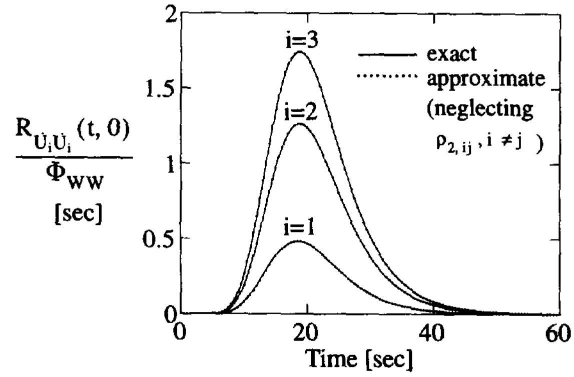

Fig. 16. Mean square relative velocity responses $\dot{U}_i(t)$ to case-V modulated white noise.

$i \neq j$). It is noted that the approximate solution is excellent. This could be expected from Fig. 11(a) where the cross-modal correlation coefficient $\rho_{0,12}(t)$ ($\omega_2/\omega_1 = 2{\cdot}73$) between the first two modes which dominate the response is shown to be very small during the strong portion of the response, namely between 10 and 40 s. Similarly, as shown in Fig. 16, the mean square relative velocity responses are accurately approximated by neglecting the cross-modal terms (i.e. assuming $\rho_{2,ij}(t) = 0$ for $i \neq j$) which are also very small (see Fig. 12(a)) for the natural frequencies of the structure considered here. Figure 17(a) indicates that the relative errors between the exact and approximate (by assuming $\rho_{a,ij}(t) = 0$ for $i \neq j$) mean square absolute acceleration responses are larger than those for the mean square relative displacement and velocity responses. This can be explained by Fig. 13(a) which shows a cross-modal correlation coefficient $\rho_{a,ij}(t)$ slightly larger than $\rho_{0,ij}(t)$ and $\rho_{2,ij}(t)$ at $\omega_2/\omega_1 = 2{\cdot}73$. Figure 17(b) shows that the approximations obtained by neglecting only the correlation terms $\rho_{1,ij}(t)$ for all i, j are very accurate. This can be explained by cancellation effects between $\rho_{1,ij}(t)$ and $\rho_{1,ji}(t)$ terms which are of different signs as seen in Fig. 14.

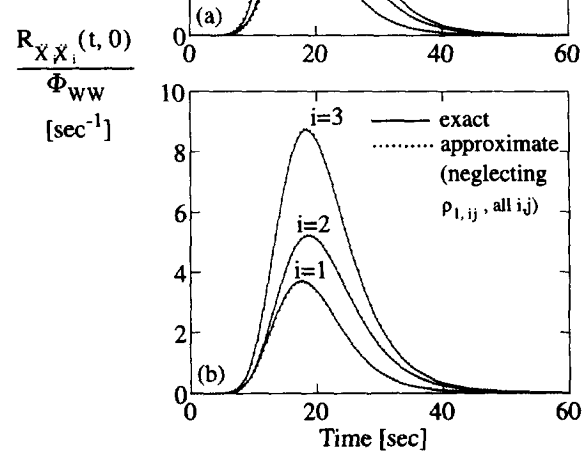

Fig. 17. Mean square absolute acceleration responses $\ddot{X}_i(t)$ to case-V modulated white noise.

CONCLUSION AND REMARKS

The analytical expressions for the correlation matrix and evolutionary power spectral density matrix of the nonstationary response of linear elastic, classically damped, MDOF systems subjected to a general nonstationary random vector process are derived using the modal superposition approach. The stationary solution for the same kind of MDOF systems excited by a stationary random vector process corresponds to a particular case of the nonstationary solution presented here. A particular emphasis is placed on the case of stochastic ground excitation and the corresponding second-order statistics of the relative displacement, velocity and absolute acceleration responses.

The main contribution of this paper consists of the explicit closed-form solutions (in terms of elementary functions) obtained for the correlation matrix and evolutionary power spectral density matrix of the response of a MDOF system excited by a modulated white noise. The explicit closed-form solution for the evolutionary power spectral density matrix is also given for the more general case of a uniformly modulated stationary process. The modulating function is the three-parameter gamma function proposed by Saragoni and Hart.[7] Modulating functions used in earlier studies, including the step and decaying exponential functions, are considered here as particular cases. Based on these explicit closed-form solutions, additional physical insight into the nonstationary behavior of linear dynamic systems can be gained.

Furthermore, the effects of cross-modal correlations on the various mean-square responses are investigated using the closed-form solutions obtained. Unlike in the stationary case, the statistical cross-modal correlation coefficients vary with time, and depend on both natural modal frequencies and not only on the modal frequency ratio ω_i/ω_j. For a fixed modal frequency ratio, they decrease as the modal frequencies increase. Also, the cross-modal correlation coefficients between modal displacement responses and modal velocity responses are not zero as in the stationary case. However, as in the stationary case, the cross-modal correlation coefficients decay fast to zero as the modal frequency ratio ω_i/ω_j (for a given ω_i or ω_j) moves away from unity (widely spaced modes), except in the early part of the excitation when the modal responses are small as observed by Xu and Igusa,[32] and they decay faster for smaller damping ratios. In general, as time progresses, they decay faster to zero as ω_i/ω_j moves away from one for a fixed ω_j. Physically speaking, when the cross-modal correlation coefficients are small, it implies that the cross-modal terms contribute little to the various mean square responses, and that statistical dependence between modal responses can be neglected.

A three-story shear building subjected to ground excitation is used to illustrate the findings of this paper.

ACKNOWLEDGEMENTS

Support from the National Science Foundation under Grant No. BCS-9210585, with Dr Shih-Chi Liu as Program Director, is gratefully acknowledged.

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